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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled SOME RESULTS ON HYPERGRAPHS submitted by CHIANG-FUNG ANDREW LIU in partial fulfilment of the requirements for the degree of Doctor or Philosophy in Mathematics.

ABSTRACT

This thesis is a study of four aspects of hypergraph theory.

In Chapter One, we give some of the basic terminology and give an overview of the remainder of the thesis, and state all of the main results.

Chapter Two is a study of various problems which arise in connection with color-critical hypergraphs. We prove first that for each pair of integers n and r, $n \geq 3$ and $r \geq 3$, there exists a critical r-chromatic linear n-graphs on m vertices for all but finitely many values of m. Secondly, we show that the number of such pairwise non-isomorphic graphs grows at least exponentially in m. Thirdly, we consider the problem of determining the least number of edges such graphs may contain and show that the number in question grows essentially linearly with m.

In Chapter Three, we consider edge-colorings of hypergraphs and obtain some new recurrence inequalities for Ramsey numbers for 3-graphs.

In Chapter Four, we consider the notion of the covering number of a hypergraph. Upper and lower bounds are obtained for these numbers and it is shown, by probabilistic methods, that there exist large classes of graphs for which the bounds given cannot be

improved by more than a constant factor.

In Chapter Five, we consider a modified coloring problem in which some, but not necessarily all, of the vertices of a hyper-graph are colored. Our main results concern the estimation of the least number of edges such a hypergraph can have.

The four topics are interrelated to some extent and these interrelations are pointed out at the appropriate places.

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TABLE OF CONTENTS

CHAPTER ONE

OVERVIEW

§1. Introduction

This thesis is a study of certain aspects of hypergraph theory. Hypergraphs, to be defined below, are a generalization of ordinary graphs which, of course, have been extensively investigated.

The present chapter is intended to give an overview of the remainder of the study. In this section, we shall give the definition of a hypergraph and introduce some of the basic concepts and terminology. In the four remaining sections, we shall introduce the specific problems in hypergraph theory which are dealt with in the thesis. All of our main results will be stated in Chapter One, the proofs being deferred to the later chapters. We attempt to minimize any confusion this may cause by restating each result when its proof is presented and numbering it the same way as in Chapter One.

We give some of the historical background to the questions being considered, but do not attempt a comprehensive survey in this regard. Instead, we mention only those results which have some direct bearing on ours.

We now give some of the basic definitions. The term being defined is underlined.

A hypergraph is an ordered pair $(V,\mathcal{F})$ where V is a finite non-empty set and $\mathcal{F}$ is a non-empty collection of subsets of V. We shall always assume that $V=\bigcup\mathcal{F}$, so that we may speak of the hypergraph $\mathcal{F}$ rather than $(V,\mathcal{F})$. The elements of $\bigcup\mathcal{F}$ are called the vertices of $\mathcal{F}$ while the elements of $\mathcal{F}$ are called the edges of $\mathcal{F}$.

A hypergraph $\mathcal{F}$ is said to be uniform if for all $F\in\mathcal{F}$, $|F|=n$ for some integer $n\geq 2$. Then $\mathcal{F}$ is called an n-graph. Thus an undirected, finite, ordinary graph without loops, multuple edges or isolated vertices is a 2-graph. We remark that 2-graph terminology not defined in this thesis can be found in [H4]. We shall deal exclusively with n-graphs. Thus the remaining definitions will be given for n-graphs, although some of them make sense for hypergraphs in general.

An n-graph $\mathcal{F}$ is said to be linear if $|F\cap F'|\leq 1$ for all $F,F'\in\mathcal{F}$, $F\neq F'$. Thus 2-graphs are necessarily linear.

Let S be a subset of the vertices of an n-graph $\mathcal{F}$. The degree of S is defined to be the number of edges of $\mathcal{F}$ which contain S. The degree of a vertex v of $\mathcal{F}$ is defined as the degree of the singleton set $\{v\}$. In the case of a 2-graph, this coincides with the usual definition of the degree of a vertex.

An n-graph $\mathcal{F}$ is said to be regular if all its vertices have degree t for some integer t. Then we say that the degree of $\mathcal{F}$ is t.

Let $\mathcal{F}$ be an n-graph with $V=\bigcup\mathcal{F}$ and $m=|V|$. $\mathcal{F}$ is said to be

complete if F consists of all n-subsets of V. We shall denote this graph by V^n or $[m]^n$.

If F' is a non-empty subset of an n-graph F, then F' is said to be a subgraph of F. If F' is a proper subset of F, then F' is said to be a proper subgraph of F.

Two n-graphs F and F' are said to be isomorphic if there exists a one-one function θ from $\bigcup F$ onto $\bigcup F'$ such that $\theta(F)\in F'$ iff $F\in F$. Otherwise F and F' are non-isomorphic.

An n-graph is said to connected if it is impossible to partition the vertex set into two disjoint non-empty sets A and B such that if F is an edge of the graph, then either $F\subset A$ or $F\subset B$. A maximal connected subgraph of an n-graph will be called a component.

Let $F_1=\{F_1^1,\ldots,F_t^1\}$ be an n_1-graph and $F_2=\{F_1^2,\ldots,F_t^2\}$ be an n_2-graph. The (n_1+n_2)-graph $F=\{F_1,\ldots,F_t\}$ where $F_i=F_i^1\cup F_i^2$, $1\leq i\leq t$, is said to be obtained by abutting F_1 to F_2. The two graphs must be disjoint.

There is one special notational convention which we shall follow throughout. The letter c will always denote a positive constant. The numerical value of c will not necessarily be the same at each occurence. Occasionally c will depend on one or more parameters which are fixed throughout the immediate discussion. When there is danger of confusion, we shall mention explicitly the parameters on which c depends. If two or more absolute constants appear in a single equation or inequality, they will be distinguished by subscripts.

Additional definitions from hypergraph theory will be

given as the need arises.

§2. Chromatic Numbers

One of the most extensively studied concepts of ordinary graph theory is that of chromatic number. We remind the reader that a 2-graph is said to have chromatic number r if r is the least positive integer for which there exists some way of coloring the vertices of the graph in r colors so that no edge joins vertices of the same color. This notion generalizes in a natural way to hypergraphs.

Formally, an n-graph $\mathcal{F}$ is said to be <u>r-colorable</u> if there exists a function $\phi:\bigcup\mathcal{F}\to\{1,\ldots,r\}$ such that $|\phi(F)|\geq 2$ for all $F\in\mathcal{F}$. We call ϕ an <u>r-coloring</u> of $\mathcal{F}$. $\mathcal{F}$ is <u>r-chromatic</u> if it is r-colorable but not $(r-1)$-colorable and we then call r the <u>chromatic number</u> of $\mathcal{F}$. $\mathcal{F}$ is <u>r-critical</u> if it is r-chromatic and all its proper subgraphs are $(r-1)$-colorable. $\mathcal{F}$ is <u>critical</u> if it is r-critical for some r.

Critical 2-graphs were first investigated by Dirac and subsequently by many other authors(see for example [O1] and references given there). It is a simple matter to verify that the only 3-critical 2-graphs are the circuits of odd length. It was hoped at one time that a characterization of the 4-critical 2-graphs would be useful in tackling the celebrated Four Color Problem.

However, no such characterization has been found and, in fact, recent results (see, for example, Simonovits([S2]) and Toft([T2])) indicate that the 4-critical 2-graphs may be quite complicated and that perhaps no simple characterization is possible.

By an (m,n,r)-graph we shall mean an r-critical n-graph on m vertices. We shall investigate various problems concerning (m,n,r)-graphs.

§2.1. The Existence Problem

There are two questions which arise:

(A) For given integers n and r, for which integers m do (m,n,r)-graphs exist?

(B) For given integers n and r, for which integers m do linear (m,n,r)-graphs exist?

For 2-graphs, and here the two problems coincide, $(m,2,3)$-graphs exist only when $m\geq 3$ is odd. This is just a restatement of the fact that the only 3-critical 2-graphs are the odd circuits. For $r\geq 4$, Dirac([D2]) proved that $(m,2,r)$-graphs exist only when $m=r$ or $m\geq r+2$.

For $n\geq 3$, let

$$M(n,r)=(n-1)(r-1)+1.$$

It is easy to verify that $[M(n,r)]^n$ is r-critical. Recently, it has been shown that (m,n,r)-graphs exist for all $m\geq M(n,r)$. This was done by Abbott and Hanson([A8]) for the case $r=3$ and by Toft([T3]) for $r\geq 4$. Since (m,n,r)-graphs do not exist for $m<M(n,r)$, problem (A) has been solved completely.

The constructions of Toft, Abbott and Hanson do not yield linear n-graphs, so that their work sheds no light on problem (B). A priori, it is not obvious whether linear (m,n,r)-graphs exist for any value of m.

The question seems to have been first raised by Gallai. Erdös and Hajnal([E6]) mention the problem and point out that the Steiner triple system on 7 points is an example of a linear (7,3,3)-graph, but give no other examples. This graph is given as Example 1 in Appendix 1.

Proofs of the existence of linear (m,n,r)-graphs for each pair of integers n and r, $n\geq 3$, $r\geq 3$, and some integer m, were given by several authors at about the same time. A simple proof based on Ramsey's Theorem(to be introduced in §3) was given by Abbott([A2]) for the case $r=3$, and his proof easily generalizes to $r\geq 4$. Other proofs were given by Erdös and Hajnal([E7]), Lovász([L4]) and Hales and Jewett([H2]). In all of these papers, the n-graphs constructed are not necessarily critical, but since any r-chromatic n-graph contains an r-critical subgraph, there is no problem.

The papers of Lovász, Erdös and Hajnal referred to above also establish the existence of arbitrarily large integers m for which linear (m,n,r)-graphs exist. A simple proof of this is given by Abbott([A7]), in the case $r=3$.

Our main result is the following

Theorem 2.1. For $n\geq 3$ and $r\geq 3$, there exists a least integer $M^*(n,r)$ such that for $m\geq M^*(n,r)$, a linear (m,n,r)-graph exists.

The determination of the numbers $M^*(n,r)$ seems to be very difficult, and we have succeeded in determining only one value. We state this as

Theorem 2.2. $M^*(3,3)=9$.

Even the next non-trivial values, $M^*(3,4)$ and $M^*(4,3)$, seem to be out of reach. The main difficulty lies in finding actual examples of linear (m,n,r)-graphs. We have been able to obtain the following bounds:

$$M^*(3,4)\leq 8928$$

and

$$M^*(4,3)\leq 62835.$$

but have no reason to believe that these are sharp.

§2.2. The Enumeration Problem

Once the existence of at least one (m,n,r)-graph has been established, it is natural to ask how many such graphs are there(to within isomorphism). We define $S(m,n,r)$ to be the number of non-isomorphic (m,n,r)-graphs and $S^*(m,n,r)$ similarly for linear (m,n,r)-graphs. Clearly

$$S(2m,2,3) = 0$$

and

$$S(2m+1,2,3) = 1.$$

Also, it is not difficult to verify that the only linear $(7,3,3)$-graph is the graph of Example 1 in Appendix 1, so that

$$S^*(7,3,3) = 1.$$

We are concerned with the case where $n \geq 3$ and $r \geq 3$ are fixed and m is large. We obtain the following lower bound for $S(m,n,r)$.

Theorem 2.3. For $n \geq 3$ and $r \geq 3$, there exists a constant $c > 1$, depending on n and r, such that

$$S(m,n,r) > c^m$$

for sufficiently large m.

For linear graphs, the situation is not quite as satisfactory as we can only prove that the lower bound holds for the cases $n=3$ and 4.

Theorem 2.4. For $n=3$ or 4 and $r \geq 3$, there exists a constant $c > 1$, depending on n and r, such that

$$S^*(m,n,r) > c^m$$

for sufficiently large m.

However, we believe that the lower bound holds for the cases $n \geq 5$. The main difficulty in supplying a proof lies again in finding actual examples of linear (m,n,r)-graphs.

§2.3. An Extremal Problem

There are numerous extremal problems that can be raised in connection with (m,n,r)-graphs and many of these have been investigated in recent years. We restrict our attention to the problem of estimating the least number of edges an (m,n,r)-graph may contain.

Put

$$E(m,n,r) = \min\{|F| : F \text{ is an } (m,n,r)\text{-graph}\}$$

and

$$E(n,r) = \min_{m} E(m,n,r).$$

Let $E^*(m,n,r)$ and $E^*(n,r)$ be similarly defined for linear (m,n,r) -graphs.

Our first result is somewhat special. It is easy to see that

$$E(2n-1,n,3) = \binom{2n-1}{n}$$

and the result

$$E(2n,n,3) = E(2n-1,n,3)$$

is implicitly contained in a paper of Abbott and Hanson([A8]). Erdös

([E5]) raised the question of evaluating $E(2n+1,n,3)$ and remarked that it does not seem easy to determine $E(9,4,3)$. We shall prove the following result:

Theorem 2.5. $23 \leq E(9,4,3) \leq 26$.

$E(9,4,3) \leq 27$ and $E(10,4,3) \leq 26$ are given in a table by Hanson([H3]) and $E(11,4,3) \leq 23$ can be deduced from a result of Toft ([T3]). The value of $E(4,3)$ is not known.

The next group of results are concerned with the behavior of $E(m,3,3)$. First we mention the following result obtained recently by Seymour([S1]):

(2.1) $$E(m,n,3) \geq m.$$

Actually, Seymour proved that in any 3-critical hypergraph, not necessarily uniform, the number of edges is at least as large as the number of vertices. We shall prove

Theorem 2.6. For $m \geq 5$,

$$E(m+3,3,3) \leq E(m,3,3)+3,$$

and for $m \geq 7$, $m \neq 8$,

$$E^*(m+3,3,3) \leq E^*(m,3,3)+3.$$

When Theorem 2.6 is combined with (2.1), we get, for $m \geq 2$, on noting that $E(7,3,3)=E^*(7,3,3)=7$,

(2.2) $$E(3m+1,3,3)=E^*(3m+1,3,3)=3m+1.$$

Theorem 2.6 and (2.2) may be used to improve Hanson's table of upper bounds for $E(m,n,3)$ in several entries.

Our remaining results are best formulated in terms of the numbers $\alpha(n,r)$ defined as follows:

$$\alpha(n,r)=\lim_{m\to\infty}\frac{E(m,n,r)}{m}$$

provided the limit exists. Let $\alpha^*(n,r)$ be defined similarly for linear graphs.

That $\alpha(2,3)$ does not exist follows from the result on the odd circuits, again. For $r\geq 4$, it is known that $\alpha(2,r)$ exists and

$$\frac{r-1}{2}+\frac{1}{2(r+9)}\leq\alpha(2,r)\leq\frac{r}{2}-\frac{1}{r-1}.$$

The lower bound is due to Gallai([G1], [G2]) and the upper bound to Hajós([H1]). It has been conjectured that the upper bound gives the correct value of $\alpha(2,r)$, but this has never been proved, even for $r=4$. That $\alpha(2,4)$ has not been evaluated is perhaps another indication that the 4-critical 2-graphs may be very difficult to characterize in any simple way.

It follows from Theorem 2.6 and (2.1) that

$$\alpha(3,3)=\alpha^*(3,3)=1.$$

Our two additional results are:

Theorem 2.7. For $n\geq 3$ and $r\geq 3$, $\alpha(n,r)$ and $\alpha^*(n,r)$ exist and are finite.

Theorem 2.8. $\alpha^*(n,r) \geq \alpha(n,r) \geq \min\{1,(r-1)/n\}$.

Theorem 2.8 will be proved by extending results of Seymour([S1]) and Dirac([D2]). It is possible that Gallai's method may be used to improve the bound, but we have not looked into this. For $n=2$ and $r\geq 4$, Theorem 2.8 is of course far weaker than the result of Gallai mentioned above.

§3. Ramsey Numbers

In this section, we consider another type of chromatic property of n-graphs, in which the objects being colored are the edges. There is, however, a relation between these edge-colorings and the vertex-colorings discussed in §2. This will be pointed out in §3.2.

Let $r\geq 2$ and $k_1,\ldots,k_r \geq n$. Let F be a complete n-graph. We attempt to color the edges of F with r colors such that F does not contain a complete subgraph $[k_i]^n$ all edges of which are colored in the ith color, $1\leq i\leq r$. If this is possible, we say that F is $(k_1,\ldots,k_r)$-colorable.

Formally, a complete n-graph F is said to be <u>$(k_1,\ldots,k_r)$-colorable</u> if there exists a function $\phi: F\to\{1,\ldots r\}$ such that for each i, $1\leq i\leq r$, all complete subgraphs $[k_i]^n$ of F contain at least one edge F for which $\phi(\mathrm{F})\neq i$. We call ϕ a <u>$(k_1,\ldots,k_r)$-coloring</u> of F.

One of the most important theorems in combinatorial mathematics is the theorem of Ramsey([R1]) which can be formulated as follows:

Ramsey's Theorem Let $r\geq 2$ and $k_1,\ldots,k_r\geq n\geq 2$. There exists a least integer $R(k_1,\ldots,k_r;n)$ such that if F is a complete n-graph with $|\cup F|>R(k_1,\ldots,k_r;n)$, then F is not $(k_1,\ldots,k_r)$-colorable.

The proof of this theorem is readily available in the literature(see for example [R4]) and is omitted.

By a $(k_1,\ldots,k_r;n)$-graph is meant a complete $(k_1,\ldots,k_r)$-colorable n-graph. A $(k_1,\ldots,k_r;n)$-graph F is maximal if $|\cup F|=R(k_1,\ldots,k_r;n)$. The numbers $R(k_1,\ldots,k_r;n)$ are called the Ramsey numbers.

A central problem has long been the evaluation or estimation of the Ramsey numbers. In particular, the case $n=2$ has been studied extensively by various authors(see [K1] and references given there). While several non-trivial Ramsey numbers for 2-graphs are known, and good bounds are available for several others, it is remarkable that not even one non-trivial Ramsey number is known for n-graphs where $n\geq 3$.

§3.1. Lower bounds for $R(k,l;3)$

In this subsection, we consider the problem of determining lower bounds for the Ramsey numbers $R(k,l;3)$. Since we are dealing

exculsively with 3-graphs, we shall suppress the parameter 3 in this subsection.

It is clear that

$$R(k,l)=R(l,k)$$

and

$$R(k,3)=k-1.$$

No non-trivial values of $R(k,l)$ have been determined. Even the value of $R(4,4)$ seems elusive. Here, it is known that

$$12\le R(4,4)\le 14. \tag{3.1}$$

The upper bound is due to Giraud([G4]) and the lower bound to Isbell([I2]). Many other authors(see [K2] and references given there) have devoted considerable effort to the problem.

The best known lower bound for $R(k,k)$ is the following due to Erdös([E2]):

$$R(k,k)>ck2^{(k^2-3k)/6}. \tag{3.2}$$

The proof of (3.2) uses non-constructive probabilistic methods. Consequently, it is of some interest to obtain lower bounds for $R(k,l)$ by means of explicit constructions, even though these bounds are weaker than the one given by (3.2).

The following results have been obtained by constructive methods. Kalbfleisch proved in [K3] that

(3.3) $$R(k,l)\geq R(k,l-1)+R(k-1,l),$$

from which it can be deduced that

(3.4) $$R(k,k)>c4^k/\sqrt{k}.$$

Abbott and Williams([A12]) proved that

(3.5) $$R(k,l)\geq 3R(k-1,l-1)$$

and

<u>Lemma 3.1.</u> For $k,l,u,v\geq 3$,

(3.6) $$R(k+u-2,l+v-2)\geq R(k,l)R(u,v).$$

Lemma 3.1 was obtained independently by Irving([I1]).

We shall prove two additional recurrence inequalities for $R(k,l)$.

<u>Theorem 3.2.</u> For $k,l,u,v\geq 3$, $k\leq v$,

(3.7) $$R(k+u-2,l+v-2)\ \geq\ \sum_{i=0}^{k-3} R(k-i,l+i)R(u+i,v-i).$$

<u>Theorem 3.3.</u> For $k\geq 4$ and $l\geq 5$,

(3.8) $$R(k,l)\geq R(k,l-2)+3R(k-1,l-1).$$

These results supercede (3.6) and (3.5). Taking $l=k+1$ in (3.8), we get

$$R(k,k+1) \geq R(k,k-1)+3R(k-1,k)=4R(k-1,k),$$

which leads to

$$R(k,k) > c4^k.$$

This is an improvement over (3.4).

Using (3.1), (3.3), (3.7), (3.8) and the bound $R(4,5) \geq 22$ of Garcia([G3]), one can obtain lower bounds for $R(k,l)$ for various small values of k and l which are superior to those appearing in the literature(see for example [A12]). We shall list some of these values in Appendix 3.

§3.2. Lower bounds for $R(k:r;3)$

We shall also consider the problem of obtaining lower bounds for $R(k_1,\ldots,k_r;3)$ for the special case where

$$k_1=\ldots=k_r=k$$

for some integer k, and we shall abbreviate $(k,\ldots,k;n)$ to $(k:r;n)$ in all contexts.

At this point, we exhibit the link between edge-colorings and vertex-colorings of n-graphs. Since we shall have occasion to refer to this connection later we call it a note.

Note Let F be a complete n-graph and let $k \geq n$. Let G be the $\binom{k}{n}$ -graph with $\bigcup G=F$ and $G=\{G_K : K \subset \bigcup F,\ |K|=k\}$ where $G_K=\{F \in F : F \subset K\}$. It is easily verified that F is $(k:r)$-colorable iff G is r-colorable, as defined in §2.

The best known lower bound for $R(k:r;3)$ is the following one, due independently to Abbott([A5]) and Irving([I1]):

(3.9) $$R(k:r;3)>exp(exp(crloglog\ k)).$$

The best known upper bound, on the other hand, is the following one due to Erdős and Rado([E9]):

(3.10) $$R(k:r;3)<exp(exp(crklog\ r)).$$

With regard to the gap between (3.9) and (3.10), we remark that it may not be easy to tighten in any significant manner the way in which the bounds depend upon r. The reason for this is that the upper bound given by (3.10) was obtained by Erdős and Rado by means of their so-called "ramification method". One of the features of this method is that any upper bound for $R(k:r;n-1)$ automatically yields an upper bound for $R(k:r;n)$. The bound given by (3.10) is obtained by their method from the following result of Skolem([S3]):

(3.11) $$R(k:r;2)<exp(crklog\ r),$$

so that the term $rlog\ r$ in (3.10) is inherited from (3.11). Now any improvement in (3.11) will effect a similar improvement in (3.10). However, there has been no significant improvement in (3.11) during the past forty years.

However, the manner in which $R(k:r;3)$ depends upon k may be sharpened considerably. Our main result is

Theorem 3.4. For $r \geq 4$ and for sufficiently large k,

$$R(k:r;3) > \exp(\exp(crk)).$$

Theorem 3.4 will in fact be proved by putting together, in the right way, a number of results which are already in the literature.

§4. Covering Numbers

A subset A of the vertex set V of a 2-graph G is called a covering set of G if every vertex in V-A is adjacent to some vertex in A.

This notion generalizes in a natural way to n-graphs. Let F be an n-graph and let $V = \bigcup F$. We say that a vertex $v \in V$ covers an $(n-1)$-subset S of V if $\{v\} \cup S \in F$. Let $A, B \subset V$. We say that A covers B if every $S \in B^{n-1}$ is covered by some $v \in A$. If A covers V-A, we say that A is a covering set of F. The covering number $\beta(F)$ of F is the minimal size of its covering sets.

We remark that the terminology in the literature is not uniform. For example, Berge([B1]) uses absorption number and Liu ([L2]) uses dominance number, while Harary([H4]) defines several types of covering numbers.

We propose to study the behavior of $\beta(F)$. It is clear that $\beta(F)$ will depend heavily on the structure of F, and in

particular on the degrees of its vertices and $(n-1)$-subsets of its vertex set. Our first main result is the following:

Theorem 4.1. Let e denote the maximal degree of the vertices of an n-graph $\mathcal{F}$ on m vertices and let g denote the minimal degree of the $(n-1)$-subsets of $\bigcup\mathcal{F}$. Let $g>0$. Then

$$\frac{\binom{m}{n-1}}{e+\binom{m-1}{n-2}} \leq \beta(\mathcal{F}) \leq \frac{m\left(1+\log\left(e+\binom{m-1}{n-2}\right)\right)}{g+n-1}. \tag{4.1}$$

Because of the many parameters involved, it may not be easy to see how the bounds for $\beta(\mathcal{F})$ given by (4.1) compare. It may therefore be worthwhile if we specialize (4.1) to the case $n=2$ and take, for definiteness, $e\sim g\sim m^{3/4}$. Thus we are considering a class of 2-graphs which are nearly regular and in which the degree of each vertex is "moderately large" but not "too large". Theorem 4.1 then shows that the covering number of any such graph $\mathcal{F}$ satisfies

$$(1+o(1))m^{1/4}<\beta(\mathcal{F})<cm^{1/4}\log m, \tag{4.2}$$

so that the bounds differ by a factor of $\log m$.

Simple examples show that the lower bound for $\beta(\mathcal{F})$ given by (4.1) cannot in general be improved. This is best seen by considering (4.2). If we take $\mathcal{F}$ to be the union of $[m^{1/4}]$ vertex disjoint copies of $[m^{3/4}]^2$, we get $\beta(\mathcal{F})=(1+o(1))m^{1/4}$.

It is natural to ask, therefore, whether the upper bound given by (4.1) can be improved. Our second main result states that, in effect, no essential improvement is possible in the cases $n=2,3,$

in the sense that there exist graphs F whose covering number $\beta(F)$ differs from the upper bound in (4.1) only by a constant factor. In particular, for the class of 2-graphs described in the paragraph following Theorem 4.1, it will follow from our results that for a large number of these graphs one has $\beta(F)>cm^{1/4}\log m$.

For $n\geq 4$, we are unable to show that the upper bound given by (4.1) is best possible. However, we shall give some limitations as to what improvements may be achieved.

We shall prove, by probabilistic methods, the existence of a class of regular n-graphs on m vertices having the properties listed in the theorems below. We call these graphs $[m,n,p]$-graphs. We do not specify p at present, but merely mention that it will be interpreted as a probability. The precise conditions which p has to satisfy will be specified in Chapter Four. In the following theorems, "almost all" means that the probability of the event in question tends to 1 as m tends to infinity. "With positive probability will mean that the probability of the event in question remains bounded away from 0 as m tends to infinity.

<u>Theorem 4.2.</u> For each $n\geq 2$ and every $\delta>0$, the degree d of almost all $[m,n,p]$-graphs satisfies

$$(1-\delta)\binom{m-1}{n-1}p\leq d\leq(1+\delta)\binom{m-1}{n-1}p.$$

<u>Theorem 4.3.</u> For each $n\geq 2$, the covering number of almost all $[m,n,p]$-graphs F satisfies

$$\beta(F) > (c \log m)/p.$$

<u>Theorem 4.4.</u> The covering number of almost all $[m,2,p]$-graphs satisfies

(4.3) $$(c_1 \log m)/p < \beta(F) < (c_2 \log m)/p.$$

We remark that the lower bound given by (4.3) together with the as yet unspecified conditions on p, coincides, in order of magnitude, with the upper bound afforded by (4.1). This shows that the upper bound given by (4.1) cannot be improved in the case of 2-graphs.

For $n=3$, the situation is not quite as satisfactory in the sense that we must replace "almost all" by "with positive probability".

<u>Theorem 4.5.</u> With positive probability the degree d of each 2-subset of the vertex set of an $[m,3,p]$-graph satisfies, for every $\delta>0$,

$$d > (1-\delta)mp.$$

<u>Theorem 4.6.</u> With positive probability, the covering number of an $[m,3,p]$-graph F satisfies

$$(c_1 \log m)/p < \beta(F) < (c_2 \log m)/p.$$

A remark similar to the one made after the statement of

Theorem 4.4 may be made here, so that for $n=3$, the upper bound for $\beta(F)$ afforded by (4.1) cannot be improved.

For $n \geq 4$, our methods break down. The difficulty seems to lie in obtaining a suitable extension of Theorem 4.5; that is, obtaining information concerning the degree of the $(n-1)$-subsets of the vertex set. We shall make a few more remarks about this after the proof of Theorem 4.5 has been given. In any case, a limitation as to how large an improvement one may hope to attain is provided by Theorem 4.3. (In terms of p, the lower bound for $\beta(F)$ provided by (4.1) is of the order of magnitude c/p.)

We remark also that the definition of covering number we have used is not the only possible one. One may for example say that a vertex covers another if both belong to the same edge, and a subset of the vertices is a covering set if every other vertex is covered by some vertex in the set. This coincides with our definition in the case $n=2$. For this problem we can prove an analogue of Theorem 4.1, and show by probabilistic methods that the upper bound for the covering number thus obtained cannot be improved in the case where n is even. Curiously, we cannot do this if n is odd, although there is no reason to believe that the result fails to hold in this case.

§5. Property B(r,s)

The final aspect of hypergraph theory which we shall consider in this thesis is a modification of the notion of chromatic number. To motivate the idea, consider the following classical theorem of van der Waerden([W1]; see also [L1]) in combinatorial number theory:

Van der Waerden's Theorem Given integers r and l, there corresponds a least integer $W=W(r,l)$ such that if $\{1,...,W\}$ is partitioned in any manner into r sets, at least one of the sets contains an arithmetic progression of length l.

A moment's reflection shows that van der Waerden's theorem is equivalent to the following: There exists a least integer W such that if G is the hypergraph with vertex set $\{1,...,W\}$ and edge set consisting of the l-subsets whose elements form an arithmetic progression, then the chromatic number of G is at least $r+1$. In fact, many classical theorems in combinatorial mathematics can be described in terms of chromatic numbers of hypergraphs(see [C1] and [H2] for a discussion of this).

Erdos([E10]) has asked whether some new light may be shed on van der Waerden's theorem by considering the following problem: Let $0<\lambda\leq 1$ and let $W=W(\lambda,r,l)$ be the least integer such that if $\{1,...,W\}$ is partitioned into r sets in an arbitrary manner, then there is an arithmetic progression of length l, at least λl terms

of which belong to the same set. What can be said about $W(\lambda,r,l)$? ([E10] appeared in 1973; the question was raised by Erdős in 1963, but the relevant paper is written in Hungarian, so we give the later reference.)

One could ask similar questions about other classical combinatorial problems. The most success in this connection has been achieved with the appropriately modified Ramsey problem(see [E11] for a statement of some of the results and further references to the literature).

This brings us then to the following definition: Let $n \geq s \geq 2$ and $r \geq 1$. An n-graph $\mathcal{F}$ will be said to have property B(r,s) if there exists a subset $S \subset \bigcup\mathcal{F}$ and a function $\phi: S \to \{1,\ldots,r\}$ such that for each i, and each $F \in \mathcal{F}$, $|\phi^{-1}(i) \cap F| \leq s-1$ and for each F, $|S \cap F| \geq 1$.

Less formally, the n-graph $\mathcal{F}$ has property B(r,s) if it is possible to color some, but not necessarily all, of the vertices of $\mathcal{F}$ in r colors so that every edge has at least one colored vertex and no edge has more than $s-1$ of its vertices the same color. If $S = \bigcup\mathcal{F}$ and $n=s$, we have the definition of r-colorable as given in §2.

Denote by $B(n,r,s)$ the least number of edges in an n-graph without property B(r,s). Evaluating $B(n,r,s)$, even for small values of the arguments, seems to be quite difficult. In our work we shall concentrate in only one aspect of the problem. We shall investigate the behavior of the function as n and s tends to infinity in such a way that the ratio n/s tends to some fixed

number $\lambda>1$. (This in the case of the van der Waerden problem is the question raised by Erdös. Unfortunately our results do not yield any new insight in this direction.) Our main result is:

Theorem 5.1. If $n_1 \geq s_1$ and $n_2 \geq s_2$, then

$$B(n_1 n_2, r, s_1 s_2) \leq B(n_1, r, s_1) B(n_2, r, s_2)^{n_1}.$$

This theorem gives some insight as to the behavior of $B(n,r,s)$ in the case where n and s tend to infinity in such a way that their ratio is essentially fixed. In the case $r=1$ we can make this a little more precise. We shall prove:

Theorem 5.2. Let $\lambda>1$ and let $n=(\lambda+o(1))s$. Then $\lim_{s\to\infty} B(n,1,s)^{1/s}$ exists.

Note Theorem 5.2 is to be understood in the following sense. There exists a number L_λ, depending only on λ, such that the following holds: Let $\varepsilon>0$. Let $Q=\langle\delta_s\rangle_{s=2}^{\infty}$ be a sequence of positive numbers converging monotonically to 0. Then there exists a number $s_1=s_1(Q,\varepsilon)$ such that if $s\geq s_1$ and $(\lambda-\delta_s)s\leq n\leq(\lambda+\delta_s)s$, then

$$(L_\lambda-\varepsilon)^s < B(n,r,s) < (L_\lambda+\varepsilon)^s.$$

We shall also consider the problem of determining $B(n,1,3)$, which we abbreviate to $B(n)$. Abbott([A1]) proved that

$$B(3n) = B(4n) = 7.$$

but was not able to evaluate $B(n)$ for any other values of n. We are not able to do this either, but we prove:

Theorem 5.3. For $n \not\equiv 0 \pmod{3 \text{ or } 4}$ and $n \neq 5, 11$,

$$8 < B(n) < 9.$$

Furthermore,

$$9 \leq B(5) \leq 11$$

and

$$8 < B(11) \leq 10.$$

CHAPTER TWO

CHROMATIC NUMBERS

§1. The Existence Problem

In this section, we present the proofs of the results stated in §2.1 of Chapter One. The proof of Theorem 2.1 is long and complicated. We try to simplify matters by splitting it up into several parts, by means of lemmas. Our first step is to extend Abbott's proof([A2]) of the existence of linear $(m,n,3)$-graphs.

Lemma 2.9. For $n\geq 3$ and $r\geq 3$, a linear (m,n,r)-graph exists for some integer m.

Proof: Let F be a complete $(n-1)$-graph with $|\cup F|>R(n:r-1;n-1)$. Let G be the n-graph with $\cup G=F$, that is, the vertices of G are the edges of F, and $G=\{G_K:K\subset\cup F,\ |K|=n\}$ where $G_K=\{F\in F:F\in K\}$. It is easy to see that G is linear. By the Note on page 16, F is $(n:r-1)$-colorable iff G is $(r-1)$-colorable. By Ramsey's Theorem, F is not $(n:r-1)$-colorable. Hence G is not $(r-1)$-colorable. It follows that G contains a linear (m,n,r)-graph as a subgraph for some integer m. This completes the proof.□

The second lemma also generalizes a result of Abbott([A7]).

Lemma 2.10. Let $n \geq 2$ and $r \geq 2$. Let l be an integer such that a linear (l,n,r)-graph G exists. For $1 \leq i \leq l$, let m_i be an integer such that a linear $(m_i,n,r+1)$-graph F_i exists. Then there exists a linear $(m,n,r+1)$-graph where $m=1+m_1+\ldots+m_l$.

Proof: Let $\cup G=\{a_1,\ldots,a_l\}$. Let $V_i=\cup F_i$ and let the V's be pairwise disjoint. Let F_i be a fixed edge of F_i and v_i a fixed vertex of F_i. Let v be a vertex such that v is not in any of the V's. Let $K_i=(F_i-\{v_i\})\cup\{v\}$.

Let F be the n-graph consisting of all the edges of F_i, $1 \leq i \leq l$, with F_i replaced by K_i, together with the edges of G with a_i replaced by v_i, so that F contains a subgraph isomorphic to G. It is easily verified that F is linear and that $|\cup F|=m$. We need show that F is $(r+1)$-critical. We do this in three steps.

Step 1. F is $(r+1)$-colorable

Since F_i is $(r+1)$-critical, $F_i-\{F_i\}$ is r-colorable. We may choose an r-coloring ψ_i such that $\psi_i(v_i)=\phi_G(a_i)$, where ϕ_G is an r-coloring of G. Define $\phi:\cup F \to \{1,\ldots,r+1\}$ by:

$$\phi(x)=\begin{cases}\psi_i(x) & \text{if } x \in V_i \\ r+1 & \text{if } x=v.\end{cases}$$

Clearly, ϕ is an $(r+1)$-coloring of F.

Step 2. $\mathcal{F}$ is not r-colorable

Suppose $\mathcal{F}$ has an r-coloring ψ. Then ψ is also an r-coloring of $\mathcal{F}_i-\{F_i\}$ for each i, $1\leq i\leq l$. Clearly we must have $\psi(v)\neq\psi(v_i)$. Thus ψ is an $(r-1)$-coloring of the n-graph obtained from G by replacing a_i by v_i. This is a contradiction since G is r-chromatic.

Step 3. All subgraphs of $\mathcal{F}$ are r-colorable

We need only consider the subgraphs of the form $\mathcal{F}-\{F\}$ for $F\in\mathcal{F}$. We consider two cases:

(i) $F=\{v_{i_1},\ldots,v_{i_l}\}$ where $G=\{a_{i_1},\ldots,a_{i_l}\}\in\mathcal{G}$

Since $\mathcal{G}$ is r-critical, $\mathcal{G}-\{G\}$ has an $(r-1)$-coloring ψ'. Let ψ_i be an r-coloring of $\mathcal{F}_i-\{F_i\}$ such that $\psi_i(v_i)=\psi'(a_i)$. Define $\psi:\bigcup\mathcal{F}\to\{1,\ldots,r\}$ by:

$$\psi(x)=\begin{cases}\psi_i(x) & \text{if } x\in V_i\\ \\ r & \text{if } x=v.\end{cases}$$

It is easily verified that ψ is an r-coloring of $\mathcal{F}-\{F\}$.

(ii) $F=K_j$ or $F\in\mathcal{F}_j-\{F_j\}$ for some j, $1\leq j\leq l$

Since $\mathcal{G}$ is r-critical, $\mathcal{G}$ has an r-coloring ϕ_G such that $\phi_G(a_j)\neq\phi_G(x)$ for any $x\in G$, $x\neq a_j$. Let ψ_i be an r-coloring of $\mathcal{F}_i-\{F_i\}$ such that $\psi_i(v_i)=\phi_G(a_i)$. Define $\psi:\bigcup\mathcal{F}\to\{1,\ldots,r\}$ by:

$$\psi(x)=\begin{cases}\psi_i(x) & \text{if } x\in V_i\\ \\ \phi_G(a_j) & \text{if } x=v.\end{cases}$$

It is easily verified that ψ is an r-coloring of $\mathcal{F}-\{F\}$.□

Lemma 2.11 Let $n \geq 3$ and let $k=2$ or 3. Let l be an integer for which there exists a linear $(l,k,3)$-graph G which is regular of degree k. For $i=1,\ldots,l$, let m_i be an integer for which a linear $(m_i,n,3)$-graph F_i exists. Then there exists a linear $(m,n,3)$-graph where $m=m_1+\ldots+m_l$.

Proof: Let $\bigcup G=\{a_1,\ldots,a_l\}$. We note first that the number of edges of G is l and that one may label the edges $G_1,\ldots,G_l$ so that $a_j \in G_j$. (This is obvious in the case $k=2$ since then G is an odd circuit. In the case $k=3$, we note that if t denotes the number of edges of G, then since each vertex appears in 3 edges and since each edge contains 3 vertices, we must have $3t=3l$ and hence $t=l$. To see that one may label the edges as indicated above, one may appeal to Hall's theorem(see for example [R4]) on systems of distinct representatives, it being easy to check that the conditions of Hall's theorem are satisfied in this case.)

Let $Q=\langle q_{ij}\rangle$ be an $l \times l$ matrix, whose entries are ordered pairs of integers, defined as follow:

$$q_{ij}=\begin{cases} (i,j) & \text{if } a_i \in G_j \\ \\ (0,0) & \text{otherwise.} \end{cases}$$

Call $(0,0)$ a zero element and the other entries non-zero elements.

Let $V_i=\bigcup F_i$ and let the V's be pairwise disjoint. Let F_i be a fixed edge of F_i and let H_i be a fixed $(k-1)$-subset of F_i. Let ζ_i be a mapping from F_i into the ith row of Q which maps F_i-H_i onto the diagonal element and maps H_i one-one onto the other non-

zero elements of the two. The mappings ζ_i induce a mapping ζ from $F_1\cup\ldots\cup F_l$ onto the non-zero elements of Q. Let

$$K_j=\{x\in F_1\cup\ldots\cup F_l:\zeta(x) \text{ is in the } j\text{th column of } Q\}.$$

Clearly each K_j is an n-set.

Let $\mathcal{F}$ be the n-graph consisting of all the edges of $\mathcal{F}_i$ with F_i replaced by K_i. It is easily verified that $\mathcal{F}$ is linear and that $|\bigcup\mathcal{F}|=m$. We need show that $\mathcal{F}$ is 3-critical. We do this in three steps.

Step 1. $\mathcal{F}$ is 3-colorable

Since $\mathcal{F}_i$ is 3-critical, $\mathcal{F}_i-\{F_i\}$ has a 2-coloring ψ_i. Define $\phi_i:\bigcup\mathcal{F}_i\to\{1,2,3\}$ by:

$$\phi_i(x)=\begin{cases}\psi_i(x) & \text{if } x\notin H_i\\ 3 & \text{if } x\in H_i.\end{cases}$$

Clearly the mappings ϕ_i induce a 3-coloring of $\mathcal{F}$.

Step 2. $\mathcal{F}$ is not 2-colorable

Suppose $\mathcal{F}$ has a 2-coloring ψ. Then ψ is also a 2-coloring of $\mathcal{F}_i-\{F_i\}$. Since $\mathcal{F}_i$ is 3-critical, $|\psi(F_i)|=1$. Define $\psi':\bigcup\mathcal{G}\to\{1,2\}$ by:

$$\psi'(a_j)=\psi(F_j)$$

for $1\leq j\leq l$. Since $\mathcal{G}$ is 3-chromatic, there exists $G_t\in\mathcal{G}$ such that $|\psi'(G_t)|=1$. It follows that $|\psi(K_t)|=1$. This is a contradiction.

Step 3. All subgraphs of $\mathcal{F}$ are 2-colorable

We need only consider the subgraphs of the form $\mathcal{F}-\{F\}$ for

$F \in \mathcal{F}$. We consider two cases:

(i) $F=K_t$ for some t, $1 \le t \le l$

Since G is 3-critical, $G-\{G_t\}$ has a 2-coloring ψ'. Let ψ_i be a 2-coloring of $\mathcal{F}_i-\{F_i\}$ such that $\psi_i(F_i)=\psi'(a_i)$. Clearly the mappings ψ_i induce a 2-coloring of $\mathcal{F}-\{F\}$.

(ii) $F \in \mathcal{F}_t-\{F_t\}$ for some t, $1 \le t \le l$

We have a 2-coloring ψ_t of $\mathcal{F}_t-\{F\}$. We treat separately the cases $k=2$ and $k=3$.

(a) $k=2$ In this case, $H_t=\{b\}$ for some b. We have a 2-coloring ψ' of $G-\{G_t\}$ such that $\psi'(G_t)=\psi_t(b)$, and a 2-coloring ψ_i of $\mathcal{F}_i-\{F_i\}$, $i \ne t$, such that $\psi_i(F_i)=\psi'(a_i)$. Clearly the mappings ψ_i induce a 2-coloring of $\mathcal{F}-\{F\}$.

(b) $k=3$ In this case, $H_t=\{b,c\}$ for some b and c. If $\psi_t(b)=\psi_t(c)$, the argument is exactly the same as in case (a). If $\psi_t(b) \ne \psi_t(c)$, we may assume that $\psi_t(b) \ne \psi_t(x)$ for some $x \in F_t-H_t$. Let $\zeta(c)$ belong to the hth column of Q. Then the argument of case (a) applies with G_t replaced by G_h. □

We need one additional result in number theory which is perhaps well-known. However, we cannot find a reference to it, so we sketch its proof.

Lemma 2.12. Let S be a set of integers such that S contains a and $a+1$ for some integer a, and for some integer $k \ge 2$, $a_1+\ldots+a_k \in S$ whenever $a_1,\ldots,a_k \in S$. Then S contains all integers greater than or equal to $(ak-a+1)a$.

Proof: Let i be an integer and let S_i denote the set of integers t satisfying $(ik-i+1)a \leq t \leq (ik-i+1)(a+1)$. Using a simple induction argument, it can be shown that $S_i \subset S$ for all $i \geq 0$ and that $S_i \cap S_{i+1} \neq \emptyset$ for $i \geq a$. The result follows immediately.□

We now prove our main result.

Theorem 2.1. For $n \geq 3$ and $r \geq 3$, there exists a least integer $M^*(n,r)$ such that for $m \geq M^*(n,r)$, a linear (m,n,r)-graph exists.

Proof: We use induction on r. We first establish the existence of $M^*(n,3)$ for fixed n. Let S be the set of integers m for which linear $(m,n,3)$-graphs exist. We need show that S contains all sufficiently large integers. We consider two cases:

(i) n is odd By Lemma 2.9, there exists an integer $m \in S$. Now an odd circuit of length n is a regular linear $(n,2,3)$-graph of degree 2. By Lemma 2.11, $a_1 + \ldots + a_n \in S$ if $a_1, \ldots, a_n \in S$. Thus $mn \in S$. Also we may consider a single n-set as a linear $(n,n,2)$-graph. It follows from the argument of Lemma 2.10 that we have $mn+1 \in S$. By Lemma 2.12, S contains all sufficiently large integers.

(ii) n is even By Lemma 2.9, there exists an integer $m \in S$. It is not difficult to verify that Example 4 in Appendix 1 is a regular linear $(22,3,3)$-graph of degree 3. By Lemma 2.11, $a_1 + \ldots + a_{22} \in S$ if $a_1, \ldots, a_{22} \in S$. Thus $22m \in S$. In Lemma 2.10, take $m_1 = 22m$ and $m_i = m$ for $i > 1$. It follows that $m(n+21)+1 \in S$. Now an odd circuit of length $21+n$ is a regular linear $(21+n,2,3)$-graph of degree 2. By Lemma 2.11,

$m(n+21)\epsilon S$. By Lemma 2.12, S contains all sufficiently large integers.

Suppose $r\geq 3$ and that $M^*(n,r)$ exists. By Lemma 2.9, there is an integer m_0 for which there exists a linear $(m_0,n,r+1)$-graph. By Lemma 2.10, there exists a linear $(m_0M^*(n,r)+1,n,r+1)$-graph. Let

(1) $$m \geq m_0(m_0+1)M^*(n,r)$$

and write

(2) $$m = qm_0+b,\ 1\leq b\leq m\ .$$

From (1) and (2), it follows easily that

(3) $$q \geq m_0M^*(n,r).$$

Let

(4) $$t = q-(b-1)M^*(n,r)$$

so that by (3) and (4),

(5) $$t \geq M^*(n,r).$$

By the induction hypothesis, (5) and the fact that $b\geq 1$, there exists a linear $(t+b-1,n,r)$-graph. In Lemma 2.10, take

(6) $$\begin{cases} m_1=\ldots=m_t=m_0 \\ m_{t+1}=\ldots=m_{t+b-1}=m\ M^*(n,r)+1. \end{cases}$$

Now by (2), (4) and (6),

$$1+m_1+\ldots+m_{t+b-1} = m$$

so that, by Lemma 2.10, a linear $(m,n,r+1)$-graph exists. This completes the proof.□

We have the following

<u>Corollary</u> Let $W(n,r)$ denote the least integer m for which a linear (m,n,r)-graph exists. Then

$$M^*(n,r+1) \leq W(n,r+1)(W(n,r+1)+1)\,M^*(n,r).$$

<u>Proof:</u> Take $m_0=W(n,r)$ in the above argument.□

<u>Theorem 2.2.</u> $M^*(3,3)=9$.

<u>Proof:</u> It is not difficult to verify that linear $(8,3,3)$-graphs do not exist. It is easy to see that Examples 1, 2 and 3 in Appendix 1 are respectively a linear $(7,3,3)$-graph, a linear $(9,3,3)$-graph and a linear $(11,3,3)$-graph. The result will follow if we can show that a linear $(m+3,3,3)$-graph exists whenever a linear $(m,3,3)$-graph exists.

Let G be a linear $(m,3,3)$-graph and let $\{v_1,v_2,v_3\}$ be a fixed edge of G. Let F be the 3-graph obtained from G by replacing $\{v_1,v_2,v_3\}$ by $\{v_1,v_2,a_3\}$, $\{v_1,a_2,v_3\}$, $\{a_1,v_2,v_3\}$ and $\{a_1,a_2,a_3\}$, where $a_1,a_2,a_3 \notin G$. It is easy to verify that F is linear and that $|\bigcup F|=m+3$. We need show that F is 3-critical. We do this in three steps.

<u>Step 1. F is 3-colorable</u>

Let ϕ_G be a 3-coloring of G. Define $\phi:\bigcup F\to\{1,2,3\}$ by:

$$\phi(x)=\begin{cases}\phi_G(x) & \text{if } x\in\bigcup G\\ \phi_G(v_i) & \text{if } x=a_i,\ 1\leq i\leq 3.\end{cases}$$

Clearly ϕ is a 3-coloring of F.

Step 2. $\mathcal{F}$ is not 2-colorable

Suppose $\mathcal{F}$ has a 2-coloring ψ. Then ψ is also a2-coloring of $\mathcal{G}-\{v_1,v_2,v_3\}$. Since $\mathcal{G}$ is 3-critical, we must have say $\psi(v_1)=\psi(v_2)=\psi(v_3)=1$. In order that the edge $\{a_1,a_2,a_3\}$be not monochromatic, we have say $\psi(a_1)=1$. Then the edge $\{a_1,v_2,v_3\}$ is monochromatic. This is a contradiction.

Step 3. All subgraphs of $\mathcal{F}$ are 2-colorable

We need only consider the subgraphs of the form $\mathcal{F}-\{F\}$ for $F\in\mathcal{F}$. We consider three cases:

(i) $F=\{a_1,a_2,a_3\}$

Since $\mathcal{G}$ is 3-critical, $\mathcal{G}-\{\{v_1,v_2,v_3\}\}$ has a 2-coloring ψ' such that $\psi'(v_1)=\psi'(v_2)=\psi'(v_3)=1$, say. Define $\psi:\bigcup\mathcal{F}\to\{1,2\}$ by:

$$\psi(x)=\begin{cases}\psi'(x) & \text{if } x\in\bigcup\mathcal{G}\\ \\ 2 & \text{if } x=a_1,\ a_2 \text{ or } a_3.\end{cases}$$

It is easily verified that ψ is a 2-coloring of $\mathcal{F}-\{F\}$.

(ii) $F=\{v_1,v_2,a_3\}$ or $\{v_1,a_2,v_3\}$ or $\{a_1,v_2,v_3\}$

We may assume that $F=\{v_1,v_2,a_3\}$. Let ψ' be a 2-coloring of $\mathcal{G}-\{\{v_1,v_2,v_3\}\}$ such that $\psi'(v_1)=\psi'(v_2)=\psi'(v_3)=1$, say. Define $\psi:\bigcup\mathcal{F}\to\{1,2\}$ by:

$$\psi(x)=\begin{cases}\psi'(x) & \text{if } x\in\bigcup\mathcal{G}\\ 2 & \text{if } x=a_1 \text{ or } a_2\\ 1 & \text{if } x=a_3.\end{cases}$$

It is easily verified that ψ is a 2-coloring of $\mathcal{F}-\{F\}$.

<u>(iii) $F \in G - \{\{v_1, v_2, v_3\}\}$</u>

Let ψ' be a 2-coloring of $G-\{F\}$. Define $\psi: \bigcup F \to \{1,2\}$ by:

$$\psi(x) = \begin{cases} \psi'(x) & \text{if } x \in \bigcup G \\ \\ \psi'(v_i) & \text{if } x = a_i,\ 1 \le i \le 3 \end{cases}$$

It is easily verified that ψ is a 2-coloring of $F-\{F\}$.□

It is easy to show that $(m,3,3)$-graphs do not exist for $m \le 6$. The existence problem in the case $n=r=3$ is thus solved completely: There exists a linear $(m,3,3)$-graph only when $m=7$ or $m \ge 9$.

We now indicate how the bounds

$$M^*(3,4) \le 8928$$

and

$$M^*(4,3) \le 62835$$

are obtained.

It has been shown by Rosa([R3]) that the 3-graph of Example 7 in Appendix 1 is 4-chromatic. Thus $W(3,4) \le 31$, where $W(n,r)$ is defined in the corollary to Theorem 2.1. By this corollary and Theorem 2.2, we have

$$M^*(3,4) \le 31(31+1)M^*(3,3) = 8928.$$

We have verified that the 4-graphs of Examples 5 and 6 in Appendix 1 contain respectively a linear $(25,4,3)$-graph and a linear $(28,4,3)$-graph. A brief indication as to how this was done is given in Appendix 2.

Let S be the set of integers m for which linear $(m,4,3)$ -graphs exist. We have $25,28\in S$. By Lemmas 2.10 and 2.11, we have

$$25+25+25+25+25+25+28 = 178\in S$$

and

$$(25+25+25+25+1)+25+25+25+1 = 177\in S.$$

By Lemma 2.11, we have $a_1+a_2+a_3\in S$ whenever $a_1,a_2,a_3\in S$. By Lemma 2.12, we have

$$M^*(4,3) \leq (3(177)-177+1)177 = 62835.$$

§2. The Enumeration Problem

In this section, we present the proofs of the results stated in §2.2 of Chapter One via several lemmas.

The first lemma is due to Dirac([D1]) for the case $n=2$, and to Toft([T3]) for the case $n\geq 3$. (Actually, Toft's proof is given for hypergraphs in general.) We shall omit the proof of this lemma.

Lemma 2.13. Let G be an r-critical n-graph, and let E be a set of $r-2$ edges of G. Then $G-E$ is connected.

We need some further terminology. An n-graph $\mathcal{F}$ is said to have property T if there exist $F\in\mathcal{F}$ and $v\in F$ such that if for any $F'\in\mathcal{F}$, $\mathcal{F}-\{F,F'\}$ is decomposed into two components $\mathcal{F}_1$ and $\mathcal{F}_2$ with say $v\in\bigcup\mathcal{F}_1$, then $|(\bigcup\mathcal{F}_1)\cap F|\geq 2$. We shall then call F a T edge and v

a T vertex.

The second lemma establishes a connectivity property of the graph constructed in Lemma 2.11, in the case $k=2$. We first of all give a less formal description of these graphs. For $k=2$, we have $l=2q+1$ for some positive integer q, and we may take

$$G = \{(1,2),\ldots,(i,i+1),\ldots,(2q,2q+1),(2q+1,1)\}.$$

Moreover, H_i is a singleton set and may therefore be identified with its only vertex, say v_i. Furthermore, it is readily seen that K_i is obtained from F_i by replacing v_i by v_{i-1} for $i=2,\ldots,2q+1$ and K_1 is obtained from F_1 by replacing v_1 by v_{2q+1}.

We remark that since for $n\geq 3$, every 3-critical n-graph has at least one vertex of degree at least 3, we may assume that F_i and v_i are chosen so that v_i has degree at least 3. We shall also suppose that at least one of the F_i has property T, choosing a T edge as F_i and a T vertex as v_i. We remark that a T vertex also has degree at least 3. Finally, we suppose that the vertex sets of the graphs F_i are as nearly equal as possible; that is, for some M, $|F_i|=M$ or $M+1$.

Lemma 2.14. Let F be the n-graph constructed via Lemma 2.11 and satisfying the added conditions given in the preceding paragraph. Let E be a set of $2q+1$ edges of F and suppose that $F-E$ consists of $2q+1$ components each of whose vertex sets is of size M or $M+1$. Then $E=\{K_1,\ldots,K_{2q+1}\}$.

Proof: Let $F_i^*=(F_i-\{F_i\})\cup\{K_i\}$ for $1\le i\le 2q+1$. Suppose $|E\cap F_j^*|=0$ for some j. F_j^* is connected by Lemma 2.13 and we have $v_{j-1}\in F_j^*$. Now all edges in F_{j-1}^* containing v_{j-1} must be removed or we shall have a component of $F-E$ of size at least $M+2$, a contradiction. By our assumption, v_{j-1} is of degree at least 2 in F_{j-1}^*. However, if $|E\cap F_{j-1}^*|\ge 2$, it follows from a simple argument that $|E|>2q+1$, which is impossible. Hence we may conclude that $K_{j-1}\notin E$. Now F_{j-1}^*-E, disregarding the isolated vertex v_{j-1}, must be connected, as otherwise $F-E$ will have a component with less than M vertices, a contradiction. However, to avoid having a component with more than $M+2$ vertices, we must have $|E\cap F_{j-2}^*|\ge 2$ as before, which in turn implies that $|E|>2q+1$. It follows that $|E\cap F_i^*|\ne 0$ for all i, and thus $|E\cap F_i^*|=1$ for all i.

Let t be such that F_t has property T. Suppose

$$E\cap F_t^* = \{F\} \ne \{K_t\}.$$

Consider $F_t-\{F_t,F\}$. If it is decomposed into two components F_1 and F_2 with say $v_t\in\bigcup F_1$, then $|(\bigcup F_1)\cap F_t|\ge 2$. On the other hand, by Lemma 2.13, $(\ F_1)\cap F_t\ne\emptyset$. Since $K_t\notin E$, F_t^*-E is connected and $F-E$ will have a component of size at least $M+2$. The same conclusion can be drawn if $F_t-\{F_t,F\}$ is connected. Hence we must have $K_t\in E$.

Suppose there exists j such that $K_j\in E$ and $K_{j+1}\notin E$. Then $F_j^*-\{K_j\}=F_j-\{F_j\}$ is connected by Lemma 2.13. Since $K_{j+1}\notin E$, $F\ E$ has a component of size at least $M+2$. The only possibility left is

$$E = \{K_1,\ldots,K_{2q+1}\}$$

as required. □

Next we need a simple number theoretic result.

Lemma 2.15. Let M be a positive integer. Then every sufficiently large integer m can be written in the form

$$m = M(q-a)+(M+1)(q+a+1) \tag{7}$$

for some positive integer q and some a, $0 \le a \le 2M+1$.

Proof: Let $m \ge 3M+2$. Then there is a unique positive integer q such that

$$Mq+(M+1)(q+1) \le m \le M(q+1)+(M+1)(q+2).$$

Thus

$$m = Mq+(M+1)(q+1)+a = M(q-a)+(M+1)(q+a+1)$$

where $0 \le a \le M(q+1)+(M+1)(q+2)-Mq-(M+1)(q+1)=2M+1$.□

The final lemma in this sequence is only one step away from our main results.

Lemma 2.16. Let $n \ge 3$ and $r \ge 3$. Suppose there exists an integer M such that there exist an $(M,n,3)$-graph and an $(M+1,n,3)$-graph one of which has property T. Then there exists a constant $c>1$, depending on n and r, such that

$$S(m,n,r) > c^m$$

for sufficiently large m. An analogous result holds for linear n-graphs.

Proof: We use induction on r. Consider first the case $r=3$.

Consider first the case $r=3$. In Lemma 2.15, take M be as given in the Theorem. Suppose that m, as given by (7), is so large that $q>2M+1$ and hence that $q-a$ is positive. Let A be a $(q-a)$-subset of $\{1,\ldots,2q+1\}$. In Lemma 2.11, take $k=2$, $l=2q+1$,

$$G = \{(1,2),\ldots,(i,i+1),\ldots,(2q,2q+1),(2q+1,1)\},$$

$m_i=M$ for $i\in$A and $m_i=M+1$ for $i\notin$A. Note that $m_1+\ldots+m_l=m$. Let $F^{(\mathrm{A})}$ be the $(m,n,3)$-graph constructed via Lemma 2.11. Let A' be another $(q-a)$-subset of $\{1,\ldots,2q+1\}$ and let $F^{(\mathrm{A}')}$ be constructed similarly. We suppose further that $F^{(\mathrm{A})}$ and $F^{(\mathrm{A}')}$ satisfy the added conditions given in the paragraph preceding Lemma 2.14.

Suppose $F^{(\mathrm{A})}$ and $F^{(\mathrm{A}')}$ are isomorphic, and let θ denote an isomorphism between them. Again using the same notation as in Lemma 2.11, let K_j and K'_j, $1\leq j\leq 2q+1$, denote the special edges of $F^{(\mathrm{A})}$ and $F^{(\mathrm{A}')}$. If we remove all of the edges K_j from $F^{(\mathrm{A})}$, we get a graph with $2q+1$ components, $q-a$ of which have M vertices and $q+a+1$ of which have $M+1$ vertices. Thus, removing the edges $\theta(\mathrm{K}_j)$ from $F^{(\mathrm{A}')}$ has the same effect. However, by Lemma 2.14, the only way this can be achieved is by deleting the edges K'_j. Thus θ must map the edges K_j onto the edges K'_j in some order. We may assume that for some t, $\theta(\mathrm{K}_t)=\mathrm{K}'_t$. By removing K_t and K_{t-1} from $F^{(\mathrm{A})}$, we obtain two components one of which is of size M or $M+1$ and containing exactly one vertex in K_t. Now, removing K'_t and $\theta(\mathrm{K}_{t-1})$ has the same effect. Thus we must have $\theta(\mathrm{K}_{t-1})=\mathrm{K}'_{t-1}$. Repeating this argument, we conclude that there is a cyclic permutation of $(1,\ldots,2q+1)$ which maps A onto A'. It follows that

$$S(m,n,3) \geq \frac{1}{2q+1}\binom{2q+1}{q-a} \geq \frac{1}{2q+1}\binom{2q+1}{q-2M-1}$$

In view of (7) and the fact that M is fixed, we get

$$S(m,n,3) > c^m$$

for some $c>1$. This completes the argument for the case $r=3$.

Let M be an integer such that an $(M,n,r+1)$-graph and an $(M+1,n,r+1)$-graph exist. Let d denote the maximum degree of the vertices of either graph. Let l max$\{2d,M(n,r)\}$. Let H be an $(M',n,r+1)$-graph$(M'=lM+1)$ constructed according to Lemma 2.10 from l copies of the $(M,n,r+1)$-graph. Let H' be similarly constructed with one copy of the $(M,n,r+1)$-graph replaced by the $(M+1,n,r+1)$-graph, whereby $|\cup H'|=M'+1$. Henceforth M' is fixed.

Let $t>M'$ be such that for some $c>1$ and all $m\geq t$,

(8) $$S(m,n,r) > c^m.$$

Henceforth t is fixed. Let $m\geq M'(t+1)$ and write

(9) $$m-1 = l'M'+b \qquad 1\leq b\leq M'.$$

Note that $l'>t$. Let F be an $(m,n,r+1)$-graph constructed according to Lemma 2.10 from $l'-b$ copies of H and b copies of H'. Let G be the (l',n,r)-graph used as the "model" in the construction. Let F' be similarly constructed using another (l',n,r)-graph G'.

Suppose F and F' are isomorphic, and let θ denote an isomorphism between them. Let v denote the "new" vertex used to construct F and $v_1,\ldots,v_{l'}$ denote the "new" vertices used to construct the graphs which are put together to form F. Let v', v'_1, $\ldots,v'_{l'}$ have analogous meaning when applied to F'. Now v is the only

vertex of F which is contained in l' edges whose pairwise intersection is $\{v\}$. Similarly v' is the only vertex of F' with the same property. Hence we must have $\theta(v)=v'$. By similar considerations, the vertices $v'_1,\ldots,v'_{l'}$ are images of $v_1,\ldots,v_{l'}$ in some order.

It follows that non-isomorphic (l',n,r)-graphs lead to non-isomorphic $(m,n,r+1)$-graphs. By (8) and (9), we have

$$S(m,n,r+1) \geq S(l',n,r) > c^{l'} > c_1^{\,m},$$

keeping in mind that M' is fixed. An analogous argument can be given for the linear graphs.□

We are now in a position to prove Theorem 2.3 and Theorem 2.4.

<u>Theorem 2.3.</u> For $n\geq 3$ and $r\geq 3$, there exists a constant $c>1$, depending on n and r, such that

$$S(m,n,r) > c^m$$

for sufficiently large m.

<u>Proof:</u> The complete n-graph on $2n-1$ vertices is clearly 3-critical and has property T, and $(2n,n,3)$-graphs exist. Hence the Theorem follows from Lemma 2.16.□

We do not give all the details in the proof of Theorem 2.4.

Theorem 2.4. For $n=3$ or 4 and $r\geq 3$, there exists a constant $c>1$, depending on n and r, such that

$$S^*(m,n,r) > c^m$$

for sufficiently large m.

Proof: The $(9,3,3)$-graph of Example 2 in Appendix 1 has property T, and $(10,3,3)$-graphs exist. By Lemma 2.16, the Theorem holds for $n=3$. The 4-graph of Example 5 in Appendix 1 is actually 3-critical and has property T. From this graph, a $(177,4,3)$-graph with property T can be constructed. Since $(178,4,3)$-graphs exist, Lemma 2.16 yields the result for $n=4$.$\square$

§3. An Extremal Problem

In this section, we present the proofs of the results stated in §2.3 of Chapter One. We recall that $E(m,n,r)$ is the minimal size of (m,n,r)-graphs.

Before we prove Theorem 2.5, we need a definition and an auxiliary lemma.

By F_n is meant the 2-graph on $\binom{2n+1}{n}$ vertices constructed as follows: F_n is the collection of n-subsets of $\{1,\ldots,2n+1\}$ and $F_n=\{\{v,v'\}:v\ v'=\emptyset\}$. Recall that$\beta(F_n)$ is the covering number of F_n, defined in §4 of Chapter One.

Lemma 2.17. For $n\geq 2$,

$$E(2n+1,n,3) \geq \beta(F_n).$$

Proof: Let F be any $(2n+1,n,3)$-graph with $\bigcup F=\{1,\ldots,2n+1\}$. Suppose $|F|<\beta(F_n)$. By the definition of F_n, F has an n-subset F which is not an edge of F and not disjoint from any edge of F. Define $\psi:\bigcup F\to\{1,2\}$ by

$$\psi(x)=\begin{cases} 1 & \text{if } x\in F \\ 2 & \text{if } x\notin F. \end{cases}$$

Clearly ψ is a 2-coloring of F. This is a contradiction.□

Theorem 2.5. $23 \leq E(9,4,3) \leq 26$.

Proof: It is not difficult, by considering F_4, to show that Example 8 in Appendix 1 is a $(9,4,3)$-graph. Hence $E(9,4,3)\leq 26$.

By Lemma 2.17, we have $E(9,4,3)\geq\beta(F_4)$. Thus we need show that $\beta(F_4)\geq 23$. We assume that this is false.

Let F be a covering set for F_4 with $|F|\leq 22$. Since

$$|\bigcup F_4| = \binom{9}{4} = 126$$

and F_4 is a regular 2-graph of degree 5, F contains at least one vertex which is not connected to any other vertices in F. We may assume that this vertex is $\langle 1,2,3,4\rangle$.

Let A=$\{1,2,3,4\}$ and B=$\{5,6,7,8,9\}$. A vertex $\langle t_1,t_2,t_3,t_4\rangle$ of F_4 is said to be of type $T_1T_2T_3T_4$ where $T_i\in\{A,B\}$, if $t_i\in T_i$ for $1\leq i\leq 4$.

Let V_1, V_2, V_3, V_4 and V_5 denote the sets of vertices of types AAAA, BBBB, AAAB, ABBB and AABB respectively. For $1 \leq i \leq 5$, let $k_i = |V_i|$. It is easily verified that $k_1=1$, $k_2=5$, $k_3=20$, $k_4=40$ and $k_5=60$.

Now the vertex $<1,2,3,4>$ covers all 5 vertices in V_2. Each vertex in V_2 covers the vertex $<1,2,3,4>$ and 4 vertices in V_3. Each vertex in V_3 covers 1 vertex in V_2 and 4 in V_4. Each vertex in V_4 covers 2 vertices in V_3 and 3 in V_5. Each vertex in V_5 covers 2 vertices in V_4 and 3 others in V_5.

For $1 \leq i \leq 5$, let $h_i = |F \cap V_i|$. By our assumption, $h_1=1$ and $h_2=0$. Let h_4 be chosen. In order to account for all vertices in V_3, we must have

$$h_3 \geq 20-2h_4.$$

In order to account for all vertices in V_5, we must have

$$4h_5 \geq 60-3h_4.$$

Now,

$$\begin{aligned} 22 \geq |F| &= h_1+h_2+h_3+h_4+h_5 \\ &\geq 1+(20-2h_4)+h_4+(60-3h_4)/4, \end{aligned}$$

from which it follows that $h_4 \geq 8$.

For $h_4=8$, we have $h_3 \geq 4$ and $h_5 \geq 9$. Since $22 \geq |F|$, we must have $h_3=4$ and $h_5=9$. However, routine verification shows that such a covering set F does not exist. For $h_4=9$, we have $h_3 \geq 2$ and $h_5 \geq 9$ with $22 \geq |F|$. It is easily verified that not all vertices in V_4 can be accounted for. The argument for $h_4 \geq 9$ is similar.□

Theorem 2.6. For $m\geq 5$,

$$E(m+3,3,3) \leq E(m,3,3)+3,$$

and for $m\geq 7$, $m\neq 8$,

$$E^*(m+3,3,3) \leq E^*(m,3,3)+3.$$

Proof: In the proof of Theorem 2.2, given a linear $(m,3,3)$-graph, we constructed a linear $(m+3,3,3)$-graph by deleting one edge and introducing four others. Three "new" vertices were introduced. Hence the result follows.□

Theorem 2.7. For $n\geq 3$ and $r\geq 3$, $\alpha(n,r)$ and $\alpha^*(n,r)$ exist.

Proof: From the proof of Lemma 2.10, we have

$$(10) \qquad E^*(m,n,r) \leq \sum_{i=1}^{l} E^*(m_i,n,r)+E^*(l,n,r-1)$$

where $m=m_1+\ldots+m_l+1$ and $E^*(l,n,2)$ is understood to be equal to 1.

By a result of Fekete([F1]), $\alpha^*(n,r)$ exists for $n\geq 3$ and $r\geq 3$. Also, an examination of the proof of Lemma 2.10 shows that (10) also holds for the function E, and hence $\alpha(n,r)$ exists also.□

Finally, we prove

Theorem 2.8. For $n\geq 3$ and $r\geq 3$,

$$\alpha^*(n,r) \geq \alpha(n,r) \geq \min\{1,(r-1)/n\}.$$

Proof: First we prove the following extension of Seymour's result. The argument is also adapted from Seymour's paper.

(11) $$E(m,n,r) \geq m.$$

Let $\mathcal{F}$ be an (m,n,r)-graph. Assume that $|\mathcal{F}|<|\bigcup\mathcal{F}|$. To each $v\in\bigcup\mathcal{F}$, associate a variable x_v. For each $F\in\mathcal{F}$, set up the equation $\Sigma x_v=0$, where the summation is taken over all vertices $v\in F$. Since $|\mathcal{F}|<|\bigcup\mathcal{F}|$, this system of equations has at least one non-trivial solution $\{u_v: v\in \mathcal{F}\}$. Let $\mathcal{F}=A\cup B\cup C$ where

$$\begin{cases} A=\{v\in\bigcup\mathcal{F}: u_v<0\} \\ B=\{v\in\bigcup\mathcal{F}: u_v=0\} \\ C=\{v\in\bigcup\mathcal{F}: u_v>0\}. \end{cases}$$

Let $\mathcal{G}$ be the subgraph of $\mathcal{F}$ which is comprised of all edges all vertices of which are in B. We may assume that $\mathcal{G}$ is non-empty. Since the solution $\{u_v: v\in\bigcup\mathcal{F}\}$ is non-trivial, $\mathcal{G}$ is a proper subgraph of $\mathcal{F}$. Since $\mathcal{F}$ is r-critical, $\mathcal{G}$ has an $(r-1)$-coloring ψ'. Define $\psi:\bigcup\mathcal{F}\to\{1,\ldots,r-1\}$ by:

$$\psi(x)=\begin{cases} \psi'(x) & \text{if } x\in\bigcup\mathcal{G} \\ 1 & \text{if } x\in A \\ 2 & \text{if } x\in C\cup B-\mathcal{G}. \end{cases}$$

Let $F\in\mathcal{F}$. If $F\in\mathcal{G}$, clearly $|\psi(F)|\geq 2$. If $F\notin\mathcal{G}$, then not all vertices in F belong to B. Since $\Sigma u_v=0$ for $v\in F$, there exist $v,v'\in F$ such that $v\in A$ and $v'\in C$. Thus $|\psi(F)|\geq 2$ also. Hence ψ is an $(r-1)$-coloring of $\mathcal{F}$, and this is a contradiction.

Our second step is to show

(12) $$E(m,n,r) \geq m(r-1)/n.$$

Let $\mathcal{F}$ be an (m,n,r)-graph. We claim that the degree of each vertex is at least $r-1$, from which (12) will follow.

Suppose there exists a vertex v of degree $t \leq r-2$. For definiteness say $v \in F_1,\ldots,F_t$. Now $\mathcal{F}-\{F_t\}$ has an $(r-1)$-coloring ψ such that $\psi(v)=r-1$. Since $t-1<r-2$, there exists an integer l, $1\leq l\leq r-2$, such that $\psi(F_i-\{v_i\})\neq l$ for $1\leq i\leq t-1$. Define $\phi:\bigcup\mathcal{F}\to\{1,\ldots,r-1\}$ by:

$$\phi(x)=\begin{cases}\psi(x) & \text{if } x\neq v\\ l & \text{if } x=v.\end{cases}$$

It is easily verified that ϕ is an $(r-1)$-coloring of $\mathcal{F}$. This is a contradiction.

The result now follows from (11) and (12).$\square$

CHAPTER THREE

RAMSEY NUMBERS

§1. Lower Bounds For $R(k,l;3)$

In this section, we present the proofs of the results stated in §3.1 of Chapter One. The proof of Lemma 3.1 is due to Abbott and Williams([A12]) which we present here since we shall have occasion to refer to the graphs constructed in it.

Lemma 3.1. For $k,l,u,v \geq 3$,

$$R(k+u-2,l+v-2) \geq R(k,l)R(u,v).$$

Proof: Let $j=R(u,v)$ and let $\mathcal{B}$ be a maximal (u,v)-graph with (u,v)-coloring ϕ'. Let $\bigcup\mathcal{B}=\{b_1,\ldots,b_j\}$. For $1\leq\lambda\leq j$, let $\mathcal{A}_\lambda$ be a maximal (k,l)-graph with (k,l)-coloring ϕ_λ, let $A_\lambda=\bigcup\mathcal{A}_\lambda$ and let the A's be pairwise disjoint. Let $V=A_1\cup\ldots\cup A_j$ and let $\mathcal{F}$ be the complete 3-graph defined on V. We need show that $\mathcal{F}$ is a $(k+u-2,l+v-2)$ -graph.

An edge $\{v_1,v_2,v_3\}$ of $\mathcal{F}$ is said to be of type xyz, where $1\leq x,y,z\leq j$, if $v_1\in A_x$, $v_2\in A_y$ and $v_3\in A_z$. Let $\langle xyz\rangle$ denote the set of all edges of $\mathcal{F}$ of type xyz. Define $\phi:\mathcal{F}\to\{1,2\}$ by:

$$\phi(F)=\begin{cases}\phi_\lambda(F) & \text{if } F\in A_\lambda\\ \phi'(\{b_x,b_y,b_z\}) & \text{if } F\in\langle xyz\rangle,\ x,y,z \text{ distinct}\\ 1 & \text{if } F\in\langle xxz\rangle,\ x>z\\ 2 & \text{if } F\in\langle xxz\rangle,\ x<z.\end{cases}$$

Let K be any complete subgraph of F such that $\phi(F)=1$ for all $F\in K$. Let $K=\bigcup K$. Since $\phi(F)=\phi'(\{b_x,b_y,b_z\})$ for $F\in\langle xyz\rangle$, x,y,z distinct, K cannot have non-empty intersection with u or more of the A's. Since $\phi(F)=2$ if $F\in\langle xxz\rangle$, $x<z$, $|K\cap A_\lambda|\leq 1$ except for at most one A_λ, and $|K\cap A_\lambda|\leq k-1$ for that A_λ. Hence

$$|K| \leq (k-1)+(u-1)-1 = k+u-3.$$

Therefore, F cannot have a complete subgraph K on $k+u-2$ vertices such that $\phi(F)=1$ for all $F\in K$. A similar argument shows that F cannot have a complete subgraph L on $l+v-2$ vertices such that $\phi(F)=2$ for all $F\in L$. This completes the proof.□

Before we prove Theorem 3.2, we give an auxiliary definition. By a (k,l,u,v)-graph is meant a complete 3-graph $F=\langle A_1,\ldots,A_j\rangle$ as constructed in Lemma 3.1.

<u>Theorem 3.2.</u> For $k,l,u,v\geq 3$, $k<v$,

$$R(k+u-2,l+v-2) \geq \sum_{i=0}^{k-3} R(k-i,l+i)R(u+i,v-i).$$

<u>Proof:</u> Let $0\leq i\leq k-3$ and let $j_i=R(u+i,v-i)$. Let $F_i=\langle A_{i1},\ldots,A_{ij_i}\rangle$ be an $(k-i,l+i,u+i,v-i)$-graph. Let $V_i=\bigcup F_i$ and let the V's be pairwise disjoint. Let $V=V_0\cup\ldots\cup V_{k-3}$ and let F be the complete

3-graph defined on V. We need show that F is a $(k+u-2,l+v-2)$-graph.

An edge $\{v_1,v_2,v_3\}$ of F is said to be of type $\alpha x\beta y\gamma z$ if $v_1\in A_{\alpha x}$, $v_2\in A_{\beta y}$ and $v_3\in A_{\gamma z}$. Let $<\alpha x\beta y\gamma z>$ denote the set of all edges of F of type $\alpha x\beta y\gamma z$. Define $\phi:F\to\{1,2\}$ by:

$$\phi(F)=\begin{cases} 2 & \text{if } F\in<\alpha x\alpha x\gamma z>,\ \alpha>\gamma \\ & \text{or } F\in<\alpha x\beta y\gamma z>,\ \alpha,\beta,\gamma \text{ distinct} \\ & \text{or } F\in<\alpha x\alpha y\gamma z>,\ \alpha<\gamma,\ x\neq y \\ 1 & \text{if } F\in<\alpha x\alpha x\gamma z>,\ \alpha<\gamma \\ & \text{or } F\in<\alpha x\alpha y\gamma z>,\ \alpha>\gamma,\ x\neq y \\ \phi_i(F) & \text{if } F\in F_i, \end{cases}$$

where ϕ_i is a $(k+u-2,l+v-2)$-coloring of F_i.

Let K be any complete subgraph of F such that $\phi(F)=1$ for all $F\in K$. Let $K=\bigcup K$. We first suppose that $K\subset F_i$ for some i, $0\leq i\leq k-3$. Then, by Lemma 3.1, $|K|<k+u-2$. Having disposed this case, we may assume that K has non-empty intersection with at least two of the V's. Since $\phi(F)=2$ if $F\in<\alpha x\beta y\gamma z>$, α,β,γ distinct, K must have non-empty intersection with exactly two V's, say V_α and V_γ, with $\alpha<\gamma$.

Since F_α is a $(k-\alpha,l+\alpha,u+\alpha,v-\alpha)$-graph, K can have non-empty intersection with $u+\alpha-1$ of the $A_{\alpha\lambda}$, $1\leq\lambda\leq j$. Since $\phi(F)=2$ for $F\in<\alpha x\alpha x\gamma z>$, $\alpha>\gamma$, we have $|K\cap A_{\alpha\lambda}|\leq 1$ for each $A_{\alpha\lambda}$. Since $\phi(F)=2$ for $F\in<\alpha x\alpha y\gamma z>$, $\alpha<\gamma$, $x\neq y$, K must have non-empty intersection with exactly one of the $A_{\gamma\lambda}$, $1\leq\lambda\leq j_\gamma$. Clearly $|K\cap A_{\gamma\lambda}|\leq k-\gamma-1$ for this $A_{\gamma\lambda}$. Hence

$$\begin{aligned}|K| = |K\cap V_\alpha|+|K\cap V_\gamma| &\leq (u+\alpha-1)+(k-\gamma-1)\\ &= (k+u-2)+(\alpha-\gamma) < k+u-2.\end{aligned}$$

Therefore, F cannot have a complete subgraph K on $k+u-2$ vertices such that $\phi(F)=1$ for all $F\in K$.

Let L be a complete subgraph of F such that $\phi(F)=2$ for all $F\in L$. Let $L=\bigcup L$. As before, we may assume that L has non-empty intersection with $V_{i_1},\ldots,V_{i_t}$, $i_1<\ldots<i_t$, where $t\geq 2$.

Since $\phi(F)=1$ for $F\in\langle\alpha x\alpha y\gamma z\rangle$, $\alpha>\gamma$, $x\neq y$, L must have, for $1\leq q\leq t-1$, non-empty intersection with exactly one $A_{i_q\lambda}$, $1\leq\lambda\leq j_{i_q}$. Since $\phi(F)=1$ for $F\in\langle\alpha x\alpha x\gamma z\rangle$, $\alpha<\gamma$, we must have, for $2\leq q\leq t$, $|L\cap A_{i_q\lambda}|\leq 1$ for each $A_{i_q\lambda}$, $1\leq\lambda\leq j_{i_q}$. Therefore $|L\cap V_{i_q}|\leq 1$ for $2\leq q\leq t-1$.

Clearly, $|L\cap V_{i_1}|\leq l+i_1-1$. Since F_{i_t} is a $(k-i_t,l+i_t,u+i_t,v-i_t)$-graph, L can have non-empty intersection with $v-i_t-1$ of the $A_{i_t\lambda}$, $1\leq\lambda\leq j_{i_t}$. Hence $|L\cap V_{i_t}|\leq v-i_t-1$. Therefore,

$$|L| = \sum_{q=1}^{t}\left|L\cap V_{i_q}\right| = \left|L\cap V_{i_1}\right| + \sum_{q=2}^{t-1}\left|L\cap V_{i_q}\right| + \left|L\cap V_{i_t}\right|$$

$$\leq (l+i_1-1)+(t-2)+(v-i_t-1)$$

$$= (l+v-2)-(i_t-i_1-t+2) < l+v-2.$$

Therefore, F cannot have a complete subgraph L on $l+v-2$ vertices such that $\phi(F)=2$ for all $F\in L$. This completes the proof.□

Theorem 3.3. For $k\geq 4$ and $l\geq 5$,

$$R(k,l) \geq R(k,l-2)+3R(k-1,l-1).$$

Proof: Let A, B and C be maximal $(k-1,l-1)$-graphs with $(k-1,l-1)$-colorings ϕ_A, ϕ_B and ϕ_C respectively. Let D be a maximal $(k.l-2)$

-graph with $(k,l-2)$-coloring $\phi_{\mathcal{D}}$. Let $A=\bigcup\mathcal{A}$, $B=\bigcup\mathcal{B}$, $C=\bigcup\mathcal{C}$ and $D=\bigcup\mathcal{D}$ and let A, B, C and D be pairwise disjoint. Let $V=A\cup B\cup C\cup D$ and let $\mathcal{F}$ be the complete 3-graph defined on V. We need show that $\mathcal{F}$ is a (k,l)-graph.

An edge $\{v_1,v_2,v_3\}$ of $\mathcal{F}$ is said to be of type $T_1T_2T_3$ where, for $1\leq i\leq 3$, $T_i\in\{A,B,C,D\}$ if $v_i\in T_i$. Let $\langle T_1T_2T_3\rangle$ denote the set of all edges of $\mathcal{F}$ of type $T_1T_2T_3$. Define $\phi:\mathcal{F}\to\{1,2\}$ by:

$$\phi(F)=\begin{cases} 2 & \text{if } F\in\langle AAB\rangle, \langle ABD\rangle, \langle ACC\rangle, \langle ACD\rangle, \\ & \quad \langle ADD\rangle, \langle BBC\rangle, \langle BCD\rangle, \langle BDD\rangle \\ & \quad \text{or } \langle CDD\rangle \\ 1 & \text{if } F\in\langle AAC\rangle, \langle AAD\rangle, \langle ABB\rangle, \langle ABC\rangle, \\ & \quad \langle BBD\rangle, \langle BCC\rangle \text{ or } \langle CCD\rangle \\ \phi_{\mathcal{T}}(F) & \text{if } F\in\mathcal{T}\in\{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}\}. \end{cases}$$

Let $\mathcal{K}$ be any complete subgraph of $\mathcal{F}$ such that $\phi(F)=1$ for all $F\in\mathcal{K}$. Let $K=\bigcup\mathcal{K}$. We may assume that $\mathcal{K}$ has at least two vertices in common with at least one of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$.

Suppose that $|K\cap A|\geq 2$. Since $\phi(F)=2$ for $F\in\langle AAB\rangle\cup\langle ACC\rangle\cup\langle ADD\rangle$, we must have $K\cap B=\emptyset$, $|K\cap C|\leq 1$ and $|K\cap D|\leq 1$. Furthermore, since $\phi(F)=2$ for $F\in\langle ACD\rangle$, K cannot have non-empty intersection with both C and D. Thus

$$|K| = |K\cap A|+|K\cap C|+|K\cap D| \leq (k-2)+1 = k-1.$$

A similar argument shows that if $|K\cap B|\geq 2$ or $|K\cap C|\geq 2$, we will also have $|K|\leq k-1$.

Suppose that $|K\cap D|\geq 2$. Since $\phi(F)=2$ for $F\in\langle ADD\rangle\cup\langle BDD\rangle\cup\langle CDD\rangle$, we must have $K\cap A=K\cap B=K\cap C=\emptyset$. Thus

$$|K| = |K\cap D| \leq k-1.$$

Therefore, $\mathcal{F}$ cannot have a complete subgraph K on k vertices such that $\phi(F)=1$ for all $F\in K$.

Let L be a complete subgraph of $\mathcal{F}$ such that $\phi(F)=2$ for all $F\in L$. Let $L=\bigcup L$. We may assume that L has at least two vertices in common with at least one of A, B, C and D.

Suppose that $|L\cap A|\geq 2$. Since $\phi(F)=1$ for $F\in \langle AAC\rangle\cup\langle AAD\rangle\cup\langle ABB\rangle$, we must have $L\cap C=L\cap D=\emptyset$ and $|L\cap B|\leq 1$. Thus

$$|L| = |L\cap A|+1 \leq (l-2)+1 = l-1.$$

A similar argument show that if $|L\cap B|\geq 2$ or $|L\cap C|\geq 2$, we will also have $|L|\leq l-1$.

Suppose that $|L\cap D|\geq 2$. Since $\phi(F)=1$ for $F\in\langle ABC\rangle$, L must be disjoint from at least one of A, B and C. We may assume that $L\cap A=\emptyset$. Since $\phi(F)=1$ for $F\in\langle BBD\rangle\cup\langle CCD\rangle$, we have $|L\cap B|\leq 1$ and $|L\cap C|\leq 1$. Thus

$$|L| = |L\cap D|+2 \leq (l-3)+2 = l-1.$$

Therefore $\mathcal{F}$ cannot have a complete subgraph L on l vertices such that $\phi(F)=1$ for all $F\in L$. This completes the proof.□

§2. Lower Bourds For $R(k:r;3)$

In this section, we present the proof of Theorem 3.4. As was mentioned in Chapter One, the proof uses a number of results scattered throughout the literature. We present these results as lemmas, and included the proofs. There are two reasons for including

the proofs. One is that it will illustrate the interplay of constructive methods on the one hand and probabilistic non-constructive methods on the other. Secondly, while Lemma 3.5 can be found in the long paper of Erdös, Hajnal and Rado([E8]), its proof is not entirely easy to extricate.

Lemma 3.5. For $r\geq 2$ and $k\geq 2$,

$$R(k+1:2r;3) \geq 2^{R(k:r;2)}.$$

Proof: Let $j=R(k:r;2)$ and let G be a maximal $(k:r;2)$-graph with $(k:r;2)$-coloring ϕ'. Let $\bigcup G=\{1,\ldots,j\}$. Let V be the set of all j-tuples of 0's and 1's and let F be the complete 3-graph defined on V. We need show that F is a $(k+1:2r;3)$-graph.

Let the vertices of F be arranged in lexicographical order, that is, $[x_1,\ldots,x_j]>[y_1,\ldots,y_j]$ if there exists an i, $1\leq i\leq j$, such that $x_i=1$, $y_i=0$ and $x_t=y_t$ for all $t<i$. Such an i is said to be the first place where $[x_1,\ldots,x_j]$ differs from $[y_1,\ldots,y_j]$.

For an edge $F=\{v_1,v_2,v_3\}$ of F, $v_1<v_2<v_3$, let $d_1(F)$ denote the first place where v_1 differs from v_2, and $d_2(F)$ the first place where v_2 differs from v_3. Since $v_1<v_2<v_3$, we have $d_1(F)\neq d_2(F)$. Thus $\{d_1(F),d_2(F)\}\in G$. Define $\phi:F\to\{1,\ldots,2r\}$ by:

$$\phi(F)=\begin{cases}\phi'(\{d_1(F),d_2(F)\}) & \text{if } d_1(F)>d_2(F)\\ \\ \phi'(\{d_1(F),d_2(F)\})+r & \text{if } d_1(F)<d_2(F).\end{cases}$$

Suppose $\mathcal{F}$ has a complete subgraph $\mathcal{K}$ such that $\phi(F)=1$ for all $F\in\mathcal{K}$. Let $K=\bigcup\mathcal{K}$ and assume that $|K|\geq k+1$. Let $K=\{v_1,\ldots,v_{k+1},\ldots\}$ with $v_1<\ldots<v_{k+1}<\ldots$. For $1\leq i\leq k$, let t_i denote the first place where v_i differs from v_{i+1}. Since $\phi(\{v_1,v_2,v_3\})=1$, we have $t_1>t_2$. Similarly, $t_2>t_3$, and so on. Thus $t_1>\ldots>t_k$. Since G is a $(k:r;2)$-graph, $\phi'(\{t_i,t_{i'}\})\neq 1$ for some i and i', $1\leq i<i'\leq k$. Consider $F=\{v_i,v_{i'},v_{i'+1}\}$. Clearly $d_1(F)=t_i$ and $d_2(F)=t_{i'}$. Thus

$$\phi(F) = \phi'(\{d_1(F),d_2(F)\}) = \phi'(\{t_i,t_{i'}\}) \neq 1.$$

This is a contradiction. Similarly, one can show that, for $2\leq i\leq 2r$, $\mathcal{F}$ does not have a complete subgraph $\mathcal{K}'$ on $k+1$ points such that $\phi(F)=i$ for all $F\in\mathcal{K}'$. This completes the proof.□

We need two additional results on 2-graphs. Lemma 3.6 is due to Abbott([A3]) and Lemma 3.7 to Erdős([E2]).

Lemma 3.6. For $r,r'\geq 2$ and $k\geq 2$,

$$R(k:r+r';2) > R(k:r;2)R(k:r';2).$$

Proof: Let $j=R(k:r';2)$ and let $\mathcal{B}$ be a maximal $(k:r';2)$-graph with $(k:r';2)$-coloring ϕ'. Let $\bigcup\mathcal{B}=\{b_1,\ldots,b_j\}$. For $1\leq i\leq j$, let $\mathcal{A}_i$ be a maximal $(k:r;2)$-graph with $(k:r;2)$-coloring ϕ_i. Let $A_i=\bigcup\mathcal{A}_i$ and let the A's be pairwise disjoint. Let $V=A_1\cup\ldots\cup A_j$ and let $\mathcal{F}$ be the complete 2-graph defined on V. We need show that $\mathcal{F}$ is a $(k:r+r';2)$-graph.

An edge $\{v_1,v_2\}$ of $\mathcal{F}$ is said to be of type xy, where $1\leq x,y\leq j$, if $v_1\in A_x$ and $v_2\in A_y$. Let $\langle xy\rangle$ denote the set of all edges of

F of type xy. Define $\phi: F \to \{1,\ldots,r+r'\}$ by:

$$\phi(F) = \begin{cases} \phi_i(F) & \text{if } F \in A_i \\ \phi'(\{b_x, b_y\})+r & \text{if } F \in \langle xy \rangle,\ x \neq y. \end{cases}$$

Let K be any monochromatic subgraph of F with $K = \bigcup K$. Suppose K has non-empty intersection with at least two of the A's, say A_x and A_y, $x \neq y$. Then for $1 \le i \le j$, $|K \cap A_i| \le 1$ as $\phi(F)=\phi_i(F) \le r$ for $F \in A_i$ while $\phi(F)=\phi'(\{b_x,b_y\}) > r$ for $F \in \langle xy \rangle$, $x \neq y$. Since B is $(k:r';2)$-colorable, K cannot have non-empty intersection with k or more of the A's. It follows that $|K| \le k-1$. On the other hand, if K has non-empty intersection with exactly one of the A's, then $|K| \le k-1$ also, since the A's are $(k:r;2)$-colorable. Therefore, F cannot have a monochromatic complete subgraph K on k vertices. This completes the proof.□

Lemma 3.7. For sufficiently large k,

$$R(k:2;2) > \exp(ck).$$

Proof: Let F be a complete 2-graph on m vertices. F has $\binom{m}{k}$ distinct complete subgraphs on k vertices. Given such a complete subgraph, F has $2^{\binom{m}{2}-\binom{k}{2}}$ subgraphs which contains it. Hence the number of subgraphs of F containing at least one complete subgraph on k vertices is at most

$$\binom{m}{k} 2^{\binom{m}{2}-\binom{k}{2}} < m^k 2^{\binom{m}{2}-\binom{k}{2}} / k!.$$

Now choose m such that $m \leqslant 2^{k/2}$. If k is sufficiently large, we have

$$2m^k < k!2^{\binom{k}{2}}.$$

It follows that less than half of the subgraphs of F contain a complete subgraph on k vertices. In other words, F has a subgraph G for which both G and its complement do not contain a complete subgraph on k vertices. Now define $\phi: F \to \{1,2\}$ by:

$$\phi(\mathrm{F}) = \begin{cases} 1 & \text{if } \mathrm{F} \in G \\ 2 & \text{if } \mathrm{F} \notin G. \end{cases}$$

It is easily seen that ϕ is a $(k:2)$-coloring of F. Thus

$$R(k:2;2) > 2^{k/2} > exp(ck)$$

for sufficiently large k. This completes the proof.□

Theorem 3.4. For $r \geq 4$ and sufficiently large k,

$$R(k:r;3) > exp(exp(crk)).$$

Proof: We may assume that r is even, say $r=2t$. By Lemma 3.7, we have

$$R(k:2;2) > exp(ck).$$

By repeated applications of Lemma 3.6, we have

$$R(k:r;2) = R(k:2t;2) > exp(ctk) > exp(c\ rk).$$

Combining this with Lemma 3.5, we have the result.□

It should be pointed out that the proof of Lemma 3.7 involves probabilistic non-constructive methods. By an explicit construction, Abbott([A4]) was able to prove that

$$R(k:r;2) > exp(cr\log k)$$

which leads to

$$R(k:r;3) > exp(exp(cr\log k)).$$

While this is weaker than Theorem 3.4, it is still an improvement over the old result

$$R(k:r;3) > exp(exp(cr\log\log k)).$$

CHAPTER FOUR

COVERING NUMBERS

In this chapter, we present the proofs of the results stated in §4 of Chapter One. We shall deduce Theorem 4.1 as a special case of a more general result.

Lemma 4.7. (General Covering Lemma)

Let A and B be non-empty finite sets. Let R be a relation from A to B such that for each $a\in A$, there exists $b\in B$ such that aRb, and for each $b\in B$, there exists $a\in A$ such that aRb. For $a\in A$, let $B_a=\{b\in B:aRb\}$ and for $b\in B$, let $A_b=\{a\in A:aRb\}$. Let $u=\min\{|A_b|:b\in B\}$ and $v=\max\{|B_a|:a\in A\}$. Let A' be a smallest subset of A such that for all $b\in B$, $b\in B_a$ for some $a\in A'$. Then

(1) $$|B|/v \le |A'| \le |A|(1+\log v)/u.$$

Proof: First we observe that

$$|A'|v \ge \Sigma|B_a| \ge |B|$$

where the summation is taken over all $a\in A'$. Hence

$$|A'| \ge |B|/v,$$

so that the lower bound in (1) is established.

We now turn our attention to the upper bound. Choose $a_1\in A$

so that $|B_{a_1}|=v$. Let $D_1=B_{a_1}$. After choosing $a_1,\ldots,a_{k-1}$, choose a_k so that $|D_k|$ is maximal where $D_k=B_{a_k}-\bigcup_{i=1}^{k-1} B_{a_i}$. The procedure terminates at say a_t, so that $B=\bigcup_{i=1}^{t} B_{a_i}$. Clearly $|A'|\le t$. We need show that

(2) $$t \le |A|(1+\log v)/u.$$

For $1\le i\le v$, let $f(i)$ be the largest integer such that $|D_{f(i)}|=i$. Let $g(v)=f(v)$ and for $1\le i\le v-1$,

$$g(i) = f(i)-f(i+1).$$

We have

(3) $$t = \sum_{i=1}^{v} g(i).$$

Let $B^v=B$ and for $1\le i\le v-1$, let $B^i=B-\bigcup_{j=1}^{f(i+1)} B_{a_j}$. Define $h(i)=|B^i|$ for $1\le i\le v$ and let $h(0)=0$. For $1\le i\le v$,

$$h(i-1) = h(i)-ig(i),$$

or equivalently,

(4) $$g(i) = (h(i)-h(i-1))/i.$$

Let $Q^i=\{(a,b)\in A\times B^i : aRb\}$ for $1\le i\le v$. Clearly

$$uh(i) \le |Q^i| \le i|A|.$$

Hence for $1\le i\le v$,

(5) $$h(i)/i \le |A|/u.$$

By (3), (4) and (5), we have

$$t = \sum_{i=1}^{v} \frac{h(i)-h(i-1)}{i}$$

$$= \frac{h(v)}{v} + \sum_{i=1}^{v-1} \frac{h(i)}{i(i+1)}$$

$$\leq \frac{|A|}{u}\left(1 + \sum_{i=1}^{v-1} \frac{1}{i+1}\right)$$

$$\leq |A|(1+\log v)/u,$$

which is (2). This completes the proof.

<u>Theorem 4.1.</u> Let e denote the maximal degree of the vertices of an n-graph F on m vertices and let g denote the minimal degree of the $(n-1)$-subsets of F. Let $g>0$. Then

$$\frac{\binom{m}{n-1}}{e+\binom{m-1}{n-2}} \leq B(F) < \frac{m\left(1+\log\left(e+\binom{m-1}{n-2}\right)\right)}{g+n-1}.$$

<u>Proof:</u> In Lemma 4.7, take $A=\bigcup F$, $B=[\bigcup F]^{n-1}$ and let aRb mean that either $a \in b$ or a covers b. It then follows that $u=g+n-1$ and $v=e+\binom{m-1}{n-2}$. The theorem follows immediately.□

We point out that we could have proved Theorem 4.1 without formulating the General Covering Lemma. However, the technique used in the proof has been employed many times during the past twenty years in a variety of situations, usually on an ad-hoc basis. We shall list some examples in Appendix 4. Recently it has been recognized that perhaps the method is a fairly fundamental one and various general formulations of it have appeared (see [C1], [S4] and [S5]). Our formulation of the result is somewhat different from

the others.

We now give the construction for the $[m,n,p]$-graphs. Let $n \geq 2$ and let $0<p<1$. Further conditions on p will be imposed as the need arises. Let m and n be relatively prime.

Let $V=\{1,\ldots,m\}$. Let the points of V be evenly distributed along a circle. Let $(v_1,\ldots,v_n)$ denote the convex n-gon determined by $\{v_1,\ldots,v_n\} \subset V$. Two n-gons are said to be congruent if they can be superimposed on each other by rotation along the circle. If $(v_1,\ldots,v_n)$ and $(v'_1,\ldots,v'_n)$ are congruent, we write

$$(v_1,\ldots,v_n) \equiv (v'_1,\ldots,v'_n).$$

The number of non-congruent n-gons is given by

(6) $$w = \binom{m}{n}/m.$$

Let $T_1,\ldots,T_w$ be any labelling of these n-gons.

Let $X_1,\ldots,X_w$ be independent random variables such that

$$\Pr[X_i=1] = p$$

and

$$\Pr[X_i=0] = 1-p$$

for $1 \leq i \leq w$. For each of the 2^w sequences $Q=\langle X_1,\ldots,X_w \rangle$, let F_Q be the n-graph with vertex set V and edge set

$$\{\{v_1,\ldots,v_n\}:(v_1,\ldots,v_n) \equiv T_i,\ X_i=1\}.$$

Such n-graphs will be called <u>$[m,n,p]$-graphs</u>.

We now establish the properties of the $[m,n,p]$-graphs. We

shall call freely upon basic facts from probability theory(see for example [F2]). In Theorems 4.2, 4.3 and 4.4, p satisfies the following condition:

$$m^{-(1/3)+\varepsilon} \le p \le \alpha \tag{7}$$

for some ε and α, $0<\varepsilon<1/3$, $0<\alpha<1$.

Theorem 4.2. For each $n\ge 2$ and every $\delta>0$, the degree d of almost all $[m,n,p]$-graphs satisfies

$$(1-\delta)\binom{m-1}{n-1}p \le d \le (1+\delta)\binom{m-1}{n-1}p.$$

Proof: The degree d of the $[m,n,p]$-graph corresponding to the sequence $\langle X_1,\ldots,X_w\rangle$ is given by

$$d = n\sum_{i=1}^{w} X_i.$$

Thus d is a random variable with mean

$$\mu = nwp = \binom{m-1}{n-1}p$$

and variance

$$\sigma^2 = n\binom{m-1}{n-1}p(1-p).$$

By Chebychev's inequality,

$$\Pr[|d-\mu|\ge\delta\mu] \le \sigma^2/\delta^2\mu^2 = n(1-p)/\delta^2\binom{m-1}{n-1}p, \tag{8}$$

and it is easy to check, using (7), that the right side of (8) tends to 0 as m tends to infinity. This proves the result.□

Theorem 4.3. For each $n \geq 2$, the covering number of almost all $[m,n,p]$-graphs F satisfies

$$\beta(F) > (c\log m)/p.$$

Proof: Let F be an $[m,n,p]$-graph corresponding to the sequence $\langle X_1, \ldots, X_w \rangle$. Let

$$(9) \qquad t = [(c\log m)/p]$$

for some constant c, to be chosen later. Let $A=\{a_1,\ldots,a_t\}$ be any t-subset of F. Let

$$V_A = \{\{v_1^1,\ldots,v_{n-1}^1\},\ldots,\{v_1^q,\ldots,v_{n-1}^q\}\}$$

be a maximal subset of $(V-A)^{n-1}$ such that all n-gons $(a_i, v_1^j, \ldots, v_{n-1}^j)$, $1 \leq i \leq t$, $1 \leq j \leq q$, are non-congruent.

For all $\{v_1,\ldots,v_{n-1}\} \notin V_A$, $\{v_1,\ldots,v_{n-1}\}$ will satisfy at least one congruence of the form

$$(10) \qquad (a_{i'}, v_1, \ldots, v_{n-1}) \equiv (a_i, v_1^j, \ldots, v_{n-1}^j)$$

where $1 \leq i, i' \leq t$ and $1 \leq j \leq q$, as otherwise we can then choose

$$\{v_1^{q+1},\ldots,v_{n-1}^{q+1}\} = \{v_1,\ldots,v_{n-1}\},$$

thereby contradiciting the maximality of V_A.

If $m \not\equiv 0 \pmod{n}$, each congruence of the form (10) has exactly n solutions. If $m \equiv 0 \pmod{n}$, one of them will have only one solution. In either case,

$$nqt^2 \geq \binom{m-t}{n-1},$$

or

$$q \geq \binom{m-t}{n-1}/nt^2 > (m-t-n+2)^{n-1}/nt^2.$$

As $t \leq (m-n+2)/2$ for sufficiently large m, so that

(11) $$q > m^{n-1}/2^n n!t^2.$$

For each i and j, $1 \leq i \leq t$, $1 \leq j \leq q$, we have

$$\Pr[a_i \text{ does not cover } \{v_1^j, \ldots, v_{n-1}^j\}]$$
$$= \Pr[X_k = 0,\ T_k \equiv (a_i, v_1^j, \ldots, v_{n-1}^j)] = 1-p.$$

Since the X's are independent,

$$\Pr[\text{A does not cover } \{v_1^j, \ldots, v_{n-1}^j\}] = (1-p)^t.$$

Thus,

$$\Pr[\text{A covers } \{v_1^j, \ldots, v_{n-1}^j\}] = 1-(1-p)^t.$$

The independence of the X's again gives

$$\Pr[\text{A covers } V_A] = (1-(1-p)^t)^q.$$

Thus

$$\Pr[\text{A covers } F] \leq (1-(1-p)^t)^q.$$

Since there are $\binom{m}{t}$ choices for A, we get

$$\Pr[\text{at least one A of size } t \text{ covers } F] \leq T$$

where

$$T = \binom{m}{t}(1-(1-p)^t)^q.$$

It remains to be shown that $T \to 0$ as $m \to \infty$. We have,

(12) $$T \le m^t(1-(1-p)^t)^q$$

$$= m^t\left[(1-(1-p)^t)^{1/(1-p)^t}\right]^{q(1-p)^t}$$

$$< m^t exp(-q(1-p)^t)$$

$$= exp(t\log m - q(1-p)^t).$$

By picking c sufficiently small, we have

(13) $$3\varepsilon > -c\log(1-\alpha)/\alpha.$$

By (7), (9) and (11), we have

(14) $$q(1-p)^t \ge (1-p)^t m^{n-1}/2^n n!t^2$$

$$\ge p^3(1-p)^{(c\log m)/p} m^{n-1}/2^n n!c^2(\log m)^2 p$$

$$\ge m^{n-2+3\varepsilon+c\log(1-p)/p}/2^n n!c^2(\log m)^2 p = Q.$$

By (7), (13) and the fact that $n\ge 2$, the numerator of Q is some positive power of m. Hence

(15) $$Q \ge 2c(\log m)^2/p = 2t\log m.$$

From (14) and (15), it follows that

$$q(1-p)^t > 2t\log m$$

for sufficiently large m. Hence, as $m\to\infty$, $(q(1-p)^t - t\log m)\to\infty$, and by (12) we have $T\to 0$ as well. This completes the proof.□

Theorem 4.4. The covering number of almost all $[m,2,p]$-graphs F satisfies

$$(c_1\log m)/p < \beta(F) < (c_2\log m)/p.$$

Proof: The lower bound is given by Theorem 4.3. The upper bound follows via some routine calculations from (7) and Theorems 4.1 and 4.2. We supress the details.□

In Theorems 4.5 and 4.6, the condition on p must be replaced by

(16) $$\gamma \leq p \leq \alpha$$

for some γ and α, $0<\gamma<\alpha<1$. Note however that if p satisfies (16), it clearly satisfies (7) so that Theorems 4.2, 4.3 and 4.4 still hold.

Theorem 4.5. With positive probability, the degree d of each 2-subset of the vertex set of an $[m,3,p]$-graph satisfies, for every $\delta>0$,

$$d > (1-\delta)mp.$$

Proof: For each k, $1\leq k\leq[m/2]$, let $Y_k=\Sigma X_i$ where the summation is taken over all i for which the 3-gon T_i contains a 2-subset $\{v,v'\}$ such that $|v-v'|=k$. It is clear that the summation consists of $m-3$ terms. Thus the mean of the random variable Y_k is

$$\mu = (m-3)p \sim mp$$

and its variance is given by

$$\sigma^2 = (m-3)p(1-p).$$

We now appeal to Chebychev's inequality and obtain

$$\Pr[|Y_k-\mu|\le\delta\mu] \ge 1-\sigma^2/\delta^2\mu^2.$$

Thus,

$$\Pr[|Y_k-\mu|\le\delta\mu \text{ for all } k,\ 1\le k\le[m/2]]$$

$$\ge (1-\sigma^2/\delta^2\mu^2)^{[m/2]}.$$

By some tedious but routine calculations, we have

$$(1-\sigma^2/\delta^2\mu^2)^{[m/2]} > exp(-(1-p)/2\delta^2p^2)$$

$$> exp(-(1-\alpha)/2\delta^2\gamma^2).$$

This shows that with positive probability

$$Y_k > (1-\delta)mp.$$

Since clearly $d\ge Y_k$, this completes the proof.□

Theorem 4.6. With positive probability, the covering number of an $[m,3,p]$-graph F satisfies

$$(c_1 log\ m)/p < \beta(F) < (c_2 log\ m)/p.$$

Proof: The lower bound follows from Theorem 4.3. The upper bound may be deduced in a straight forward manner from (16), Theorems 4.1, 4.2 and 4.5.□

We conclude this chapter with two remarks. First, concerning possible improvements, for $n\ge4$, of the upper bound given by Theorem 4.1, the main difficulty seems to lie in getting estimates for the degree of the $(n-1)$-subsets of the vertex set, that is, a suitable extension of Theorem 4.5. If one tries to model a proof

after the one given for $r=3$, a difficulty arises at the last step in that the probability in question cannot be shown to be bounded away from 0 as $m \to \infty$. In this connection, we remark that for the problem at hand, better tools than Chebychev's inequality are available for estimating the probabilities involved, but these do not seem to work either.

Secondly, the probabilistic arguments used here may be applied in other situations. In the course of doing this work, we have found that the methods can be applied successfully in connection with a recent problem of Stein([S5]) on quasigroups. We do not elaborate on this any further here, since the problem itself is somewhat remote from hypergraph theory.

CHAPTER FIVE

PROPERTY B(r,s)

In this chapter, we present the proofs of the results stated in §5 of Chapter One. We recall that $B(n,r,s)$ is the minimal size of n-graphs without property B(r,s).

Theorem 5.1. For $n_1 \geq s_1$ and $n_2 \geq s_2$,

$$B(n_1 n_2, r, s_1 s_2) \leq B(n_1, r, s_1) B(n_2, r, s_2)^{n_1}.$$

Proof: Let G be an n_1-graph without property B(r,s_1). Let $\bigcup G = \{v_1, \ldots, v_m\}$. For $1 \leq i \leq m$, let F_i be an n_2-graph without property B(r,s_2). Let $V_i = \bigcup F_i$ and suppose the V's are pairwise disjoint. Let G be an edge of G. For each $v_i \in G$, select an edge from F_i. The union of these edges is an $n_1 n_2$-set. Let F be the $n_1 n_2$-graph consisting of all $n_1 n_2$-sets constructed in this manner. Clearly

$$|F| = B(n_1, r, s_1) B(n_2, r, s_2)^{n_1}.$$

We shall now prove that F does not have property B(r,$s_1 s_2$).

Suppose there exists a subset $S \subset \bigcup F$ and a function $\phi : S \to \{1, \ldots, r\}$ such that

$$|\phi^{-1}(j) \cap F| \leq s_1 s_2 - 1 \quad \text{for all } j \text{ and for all } F \in F$$

and

(1) $\quad |S \cap F| \geq 1 \quad$ for all $F \in \mathcal{F}$.

We wish to arrive at a contradiction.

Let $T=\{v_i : S \cap F \neq \emptyset$ for all $F \in \mathcal{F}_i\}$. For each $\mathcal{F}_i$ such that $v_i \in T$, we must have

(2) $\quad |\phi^{-1}(j_i) \cap F_i| \geq s_2$

for some $j_i \in \{1,\ldots,r\}$ and some $F_i \in \mathcal{F}_i$, as $\mathcal{F}_i$ does not have property $B(r,s_2)$. Note that the choices of j_i and F_i need not be unique, but once chosen they are fixed. Define $\theta : T \to \{1,\ldots,r\}$ by $\theta(v_i)=j_i$.

Suppose $T \cap G=\emptyset$ for some $G \in \mathcal{G}$. Then it follows easily that $S \cap F=\emptyset$ for some $F \in \mathcal{F}$, contrary to (1). Hence $T \cap G \neq \emptyset$ for all $G \in \mathcal{G}$. Since $\mathcal{G}$ does not have property $B(r,s_1)$, for some j $\{1,\ldots,r\}$ and some $G \in \mathcal{G}$, we have

(3) $\quad |\phi^{-1}(j) \cap G| \geq s_1$.

By (2) and (3), keeping in mind the definitons of the things

involved, we see that there must exist an edge F of $\mathcal{F}$ such that

$$|\phi^{-1}(j)\cap F| \le s_1 s_2.$$

Thus $\mathcal{F}$ does not have property $B(r,s_1 s_2)$.□

Before proving Theorem 5.2, we give some lemmas, although only special cases of these are needed in the proof of Theorem 5.2.

Lemma 5.4. Let m and e be positive integers for which there exists an n-graph H on m vertices and having e edges and which is not r-colorable. Then

$$B(n,r,s) \le mB(n-1,r,s-1)+e.$$

Proof: Let $j=B(n-1,r,s-1)$ and let $G=\{G_1,\ldots,G_j\}$ be a minimal $(n-1)$-graph without property $B(r,s-1)$. Let $\bigcup H=\{a_1,\ldots,a_m\}$. Let $F^i_t=G_t\cup\{a_i\}$ and let $F_i=\{F^i_1,\ldots,F^i_j\}$. Let $\mathcal{F}$ be the n-graph whose edge set consists of all of the edges of each F_i together with the edges of H. Clearly

$$|\mathcal{F}| = mB(n-1,r,s-1)+e.$$

We shall show that $\mathcal{F}$ does not have property $B(r,s)$.

Suppose there exists a subset S of $\bigcup\mathcal{F}$ and a function $\phi:S\to\{1,\ldots,r\}$ such that

$$|\phi^{-1}(i)\cap F| \le s-1 \qquad \text{for all } i \text{ and all } F\in\mathcal{F}$$

and

$$(4) \qquad |S\cap F| \le 1 \qquad \text{for all } F\in\mathcal{F}.$$

Write $S=S_1\cup S_2$ where $S_1\subset\bigcup G$ and $S_2\subset\bigcup H$. Note that if $S_2=\bigcup H$, then ϕ is an r-coloring of H, which is a contradiction. Hence we assume $S_2\ne\bigcup H$.

Suppose for definiteness $S_2=\{a_1,\ldots,a_q\}$ where $q<m$. Then ϕ is an r-coloring of F_{q+1}. Thus there exists $t\in\{1,\ldots,r\}$ such that

$$|\phi^{-1}(t)\cap F| \geq s-1$$

for some $F\in F_{q+1}$. Now if $\phi(a_l)=t$ for some l, $1\leq l\leq q$, we have

$$|\phi^{-1}(t)\cap(F\cup\{a_l\})| \geq s.$$

Hence we may suppose that $\phi(a_i)\neq t$ for $i=1,\ldots,q$. Now define a function θ: $H\to\{1,\ldots,r\}$ by:

$$\theta(a_i)=\begin{cases}\phi(a_i) & \text{for } 1\leq i\leq q\\ \\ t & \text{for } q+1\leq i\leq m.\end{cases}$$

Then since H has no r-coloring, $\theta^{-1}(t)\supset H$ for some $H\in H$. Thus $\phi^{-1}(t)\cap H=\emptyset$, contradicting (4).□

Corollary $\quad B(n,1,s) \leq (n+1)B(n-1,1,s-1)$.

Proof: Take H to be a single n-set in Lemma 5.4. This gives the result as $B(n-1,1,s-1)>1$.□

Lemma 5.5. $\quad B(n,r,s) \leq B(n,r,s+1)$

and

$$B(n,r,s) \leq B(n+1,r,s+1).$$

Proof: This first inequality is obvious. The second one follows from the observation that if F is an $(n+1)$-graph not possessing property $B(r,s+1)$ then the n-graph obtained from F by deleting an arbitrary vertex from each edge does not have property $B(r,s)$.□

We now have sufficient information at our disposal to prove Theorem 5.2.

Theorem 5.2. Let $\lambda>1$ and let $n=(\lambda+o(1))s$. Then $\lim_{s\to\infty} B(n,1,s)^{1/s}$ exists.

Proof: We remind the reader that Theorem 5.2 is to be interpreted as in the Note on page 25. Because of the many parameters involved, the details of the proof of Theorem 5.2 are rather intricate, so we present the proof only for the case $n=[\lambda s]$. Most of the essential features of the complete proof are contained in the proof of this special case. We shall also write $B(n,1,s)$ as $B(n,s)$.

Let $\varepsilon>0$ be given and let $\alpha=\underline{\lim}_{s\to\infty} B([\lambda s],s)^{1/s}$. We shall show that

$$B([\lambda s],s)^{1/s} < (1+\varepsilon)^2(\alpha+\varepsilon)$$

for all sufficiently large s. The result will then follow.

Choose t such that $B([\lambda t],t)^{1/t}<\alpha+\varepsilon$ and $(4[\lambda t]+8)^{1/t}<1+\varepsilon$. For s sufficiently large, write $s=qt+b$ with $0\le b\le t-1$. Consider first the special case where $b=0$. Let $u=[q\lambda t]-q[\lambda t]\le q$. We have

$$\begin{aligned}
B([\lambda s],s) &= B(q[\lambda t]+u,qt) \\
&\le B(q[\lambda t]+u,qt+u) \\
&\le B(q[\lambda t]+q,qt+q) \\
&\le B(q,q)B([\lambda t]+1,t+1)^q \\
&\le B(q,q)([\lambda t]+2)^q B([\lambda t],t)^q \\
&\le 4^q([\lambda t]+2)^q B([\lambda t],t)^q
\end{aligned}$$

by Theorem 5.1, Lemma 5.5 and the corollary to Lemma 5.4. It follows that

$$B([\lambda s],s)^{1/s} < (4[\lambda t]+8)^{1/t}B([\lambda t],t)^{1/t}$$
$$< (1+\varepsilon)(\alpha+\varepsilon).$$

For the general case, let $v=[q\lambda t+\lambda b]-[q\lambda t]\geq b$. Thus

$$B([\lambda s],s) = B([q\lambda t]+v,qt+b)$$
$$\leq B([q\lambda t]+v,qt+v)$$
$$\leq ([q\lambda t]+v+2)^{v}B([q\lambda t],qt).$$

Hence,

$$B([\lambda s],s)^{1/s} \leq ([q\lambda t]+v+2)^{\frac{v}{qt+b}}B([q\lambda t],qt)^{\frac{1}{qt+b}}$$
$$< ([q\lambda t]+v+2)^{\frac{t+1}{qt+b}}((1+\varepsilon)(\alpha+\varepsilon))^{\frac{qt}{qt+b}}$$
$$< (1+\varepsilon)^{2}(\alpha+\varepsilon)$$

for sufficiently large s. This completes the proof.□

Before we prove Theorem 5.3, we need a definition and an auxiliary lemma. A subset S of the vertex set of an n-graph F is said to be an $(n,1,s)$-set if $1\leq|S\cap F|\leq s-1$ for all $F\in F$. Thus an n-graph with property B$(1,s)$ will have at least one $(n,1,s)$-set.

Lemma 5.6. Let $n\geq s\geq 3$. Let F be an n-graph without property B$(1,s)$. Let x,y $\in F$ be of degree x and y respectively. If $x+y>|F|$, then x and y appear together in at least $x+y-|F|+1$ edges. If $x+y=|F|$, then x and y appear together in at least 2 edges.

Proof: The first assertion is trivial as otherwise {x,y} is an

$(n,1,s)$-set of $\mathcal{F}$. Suppose $x+y=|F|$. Now we have $x,y\in F$ for some $F\in\mathcal{F}$. Suppose they appear together in F only. Then there exists $F'\in\mathcal{F}$ such that $x,y\notin F'$. Let $z\in F'-F$. Then $\{x,y,z\}$ is an $(n,1,s)$-set of $\mathcal{F}$. This is a contradiction and thus the second assertion is also true.□

Theorem 5.3. For $n\not\equiv 0$ (mod 3 or 4) and $n\neq 5,11$,

$$8 \leq B(n) \leq 9.$$

Furthermore,

$$9 \leq B(5) \leq 11 \tag{5}$$

and

$$8 \leq B(11) \leq 10. \tag{6}$$

Proof: We first establish the upper bounds. It is easy to verify that the upper bounds in (5) and (6) follow respectively from Examples 9 and 11 in Appendix 1. Also, Example 10 in Appendix 1 shows that

$$B(7) \leq 9.$$

By abutting two disjoint copies of this 7-graph, we have a 14-graph which shows that

$$B(14) \leq 9.$$

To complete the argument for the upper bounds, we need show that

$$B(n+3) \leq B(n)$$

for $n\not\equiv 0$(mod 3). Abut a suitable number of disjoint copies of the 3-graph of Example 12 in Appendix 1 to either the 7-graph or the 14-graph indicated above. It is routine to verify that the resulting n-graph does not have property B($1,3$). Hence

$$B(n) \leq 9$$

for $n \neq 5, 11$ as required.

Now we establish the lower bounds, but deferring the proof of $B(5) \geq 9$, which is quite involved, to Appendix 5. Throughout the remainder of the proof, Lemma 5.6 will be appealed to frequently and without explicit reference.

Let $n \not\equiv 0$ (mod 3 or 4). Suppose $F = \{F_1, \ldots, F_7\}$ is an n-graph without property B($1,3$). We may assume that every pair of vertices of F meet at least once, that is, appear together in at least one edge of F. It is easy to see that F has no vertex of degree 1, 6 or 7. We consider two cases:

(i) F has a vertex x of degree 5

Let $x \in F_1 \cap \ldots \cap F_5$. Suppose there exists $y \in F$ of degree 2. Then x meets y twice. Clearly, the vertices in F_6 are distinct from those in F_7. Now y cannot meet every vertex in $F_6 \cup F_7$. This is a contradiction. Hence there is no vertex of degree 2. Let A, B and C denote respectively the set of vertices of degree 5, 4 and 3. Let $a = |A|$, $b = |B|$ and $c = |C|$. Then

(7) $$5a + 4b + 3c = 7n.$$

We consider two subcases:

(a) $B \neq \emptyset$ Let $y \in B$. Then y must meet every vertex of A three times and every other vertex twice. Hence we have

(8) $$4(n-1) \geq 3a + 2(b-1) + 2c.$$

Now x must meet every other vertex of A four times, every vertex of B three times and every vertex of C twice. Thus

(9) $$5(n-1) \geq 4(a-1)+3b+2c.$$

From (7), (8) and (9), we have

$$14n-14 \geq 14n+a-10$$

which is a contradiction.

<u>(b) $B=\emptyset$</u> Let $F_6=\{a_1,\ldots,a_n\}$ and $F_7=\{b_1,\ldots,b_n\}$. Suppose for some i and j, we have $a_i,b_j \in A$. Then a_i and b_j must meet four times. However, a_i must meet each $b \in F_7$ twice and b_j must meet each $a \in F_6$ twice. This is impossible. Hence we may assume that $F_7 \subset C$. On the other hand, we cannot have $F_6 \subset A$ as otherwise every $a \in F_6$ must appear in each of two of $F_1,\ldots,F_5$. Hence we may assume that $a_1,\ldots,a_t \in C$ and $a_{t+1},\ldots,a_n \in A$ where $t<n$.

Let $a \in F_6 \cap C$ and $b \in F_7$. We claim that they cannot meet twice. Suppose $a,b \in F_1 \cap F_2$. Now a must meet every vertex in F_7 and b must meet every vertex in F_6. These meetings must occur in F_1 or F_2. This is impossible.

We may assume that $|F_1 \cap F_6 \cap C| \geq |F_i \cap F_6 \cap C|$ for $2 \leq i \leq 5$. Clearly F_1 must contain some $b \in F_7$. Now b must meet all of $a_{t+1},\ldots,a_n$ at least twice. Hence we must have $a_{t+1},\ldots,a_n \in F_1$. Therefore, F_1 cannot contain all of $a_1,\ldots,a_t$, say $a_t \notin F_1$. We may assume that $a_t \in F_2 \cap F_3$.

Not all vertices in $F_1 \cap F_6 \cap C$ can occur in F_2 or F_3 by the maximality assumption on F_1. Suppose we have $a_1 \in F_1-(F_2 \cup F_3)$, say $a_1 \in F_4$. Now $F_1 \cup F_4$ must contain every $b \in F_7$ just once Similarly, $F_2 \cup F_3$ must contain every $b \in F_7$ exactly once. Now each $a \in F_6 \cap A$ meets each $b \in F_7$ twice. These meetings must occur in $F_1,\ldots,F_4$. This is

impossible.

On the other hand, suppose we have $a_1 \epsilon F_1 \cap F_2$. Then we must have $a_2 \epsilon F_1 \cap F_3$ where $a_2 \epsilon C$. Now $F_1 \cup F_2$ must contain every $b \epsilon F_7$ exactly once. Similarly, $F_1 \cup F_3$ must contain every $b \epsilon F_7$ exactly once. Clearly $F_7 \cap F_2 = F_7 \cap F_3 \neq \emptyset$. Then a_t will meet some $b \epsilon F_7$ twice. This is impossible.

(ii) F has no vertex of degree 5

Let A, B and C denote respectively the set of vertices of degree 4, 3 and 2. Let $a=|A|$, $b=|B|$ and $c=|C|$. Then

(10) $$4a+3b+2c = 7n.$$

Suppose $A=\emptyset$. Since $n \not\equiv 0 \pmod 3$, $C \neq \emptyset$. Let $x \epsilon C$. Now there are at least $(7n-2)/3$ other vertices in F. As $n>4$, x cannot meet all of them. Thus $A \neq \emptyset$. We consider two subcases:

(a) $B \neq \emptyset$ Let $y \epsilon B$. Then y must meet every vertex in A twice and every other vertex once. Hence we have

(11) $$3(n-1) \geq 2a+(b-1)+c.$$

Since $A \neq \emptyset$, let $x \epsilon A$. Now x must meet every vertex in A or B twice and every vertex in C once. Thus

(12) $$4(n-1) \geq 2(a-1)+2b+c.$$

From (10), (11) and (12), we have

$$7n-7 \geq 7n-3$$

which is a contradiction.

(b) $B=\emptyset$ Since $n \not\equiv 0 \pmod 4$, $C \neq \emptyset$. Define a $(|\cup F|-n)$-graph $G=\{G_1,\ldots,G_7\}$ by $G_i = F-F_i$ for $1 \leq i \leq 7$. Now every vertex in A has degree 3 in G and every vertex in C has degree 5 in G. By subcase 1(b), G has a

$(|\bigcup F|-n,1,3)$-set S. It is easy to see that $|S|=2$, 3 or 4.

Suppose $|S|=2$, say $S=\{x,y\}$. Now for all $G\in\mathcal{G}$, $x\in G$ or $y\in G$. Hence x and y do not meet in F. This is a contradiction. Suppose $|S|=3$. For $1\leq i\leq 7$, $1\leq|S\cap G_i|\leq 2$. Hence $2\geq|S\cap F_i|\geq 1$. Thus S is an $(n,1,3)$-set of F. This is a contradiction. Finally, suppose $|S|=4$. Then either S contains four vertices in A or three vertices in A and one vertex in C. In either case, it is easy to show that one vertex is superfluous, and we may choose S such that $|S|=3$. This has been shown to be impossible. This completes the proof.□

APPENDIX 1

EXAMPLES OF n-GRAPHS

We list here the examples of n-graphs which we refer to in our study. The vertex set in each example is taken to be $\{1,\ldots,m\}$ for some integer m and we shall simply list the edge set.

Example 1

$\{1,2,3\}$ $\{1,4,5\}$ $\{1,6,7\}$ $\{2,4,6\}$ $\{2,5,7\}$ $\{3,4,7\}$ $\{3,5,6\}$

This 3-graph is a Steiner triple system with parameters $(7,7,3,3,1)$.

Example 2

$\{1,4,7\}$ $\{1,5,9\}$ $\{1,6,8\}$ $\{2,3,8\}$ $\{2,4,6\}$ $\{2,7,9\}$ $\{3,4,5\}$ $\{3,6,9\}$

$\{4,8,9\}$ $\{5,7,8\}$

This 3-graph is our own construction.

Example 3

$\{1,2,3\}$ $\{1,4,9\}$ $\{1,5,10\}$ $\{1,6,11\}$ $\{2,5,9\}$ $\{2,6,10\}$ $\{2,7,11\}$ $\{3,6,9\}$

$\{3,7,10\}$ $\{3,8,11\}$ $\{4,5,6\}$ $\{6,7,8\}$ $\{9,10,11\}$

This 3-graph is our own construction.

Example 4

$\{1,4,5\}$ $\{1,6,7\}$ $\{1,8,15\}$ $\{2,3,22\}$ $\{2,4,6\}$ $\{2,5,7\}$ $\{3,4,7\}$ $\{3,5,6\}$

{8,11,12} {8,13,14} {9,10,22} {9,11,13} {9,12,14} {10,11,14}

{10,12,13} {15,18,19} {15,20,21} {16,17,22} {16,18,20} {16,19,21}

{17,18,21} {17,19,20}

This 3-graph is constructed from Example 1 according to Lemma 2.11, using an odd circuit of length 3 as the basis.

Example 5

{1,2,6,25} {1,3,11,19} {1,4,14,17} {1,5,10,24} {1,7,8,12}

{1,9,16,18} {1,13,15,23} {1,20,21,22} {2,3,7,21} {2,4,12,20}

{2,5,15,18} {2,8,9,13} {2,10,17,19} {2,11,14,24} {2,16,22,23}

{3,4,8,22} {3,5,13,16} {3,6,18,20} {3,9,10,14} {3,12,15,25}

{3,17,23,24} {4,5,9,23} {4,6,10,15} {4,7,16,19} {4,11,13,21}

{4,18,24,25} {5,6,7,11} {5,8,17,20} {5,12,14,22} {5,19,21,25}

{6,8,16,24} {6,9,19,22} {6,12,13,17} {6,14,21,23} {7,9,17,25}

{7,10,20,23} {7,13,14,18} {7,15,22,24} {8,10,18,21} {8,11,23,25}

{8,14,15,19} {9,11,15,20} {9,12,21,24} {10,11,12,16} {10,13,22,25}

{11,17,18,22} {12,18,19,23} {13,19,20,24} {14,16,20,25} {15,16,17,21}

This 4-graph is a block design with parameters (25,50,8,4,1).

Example 6

{1,2,11,20} {1,3,12,21} {1,4,12,22} {1,5,14,23} {1,6,15,24}

{1,7,16,25} {1,8,17,26} {1,9,18,27} {1,10,19,28} {2,3,25,28}

{2,4,24,27} {2,5,18,19} {2,6,21,23} {2,7,13,14} {2,8,15,16}

{2,9,12,17} {2,10,22,26} {3,4,23,26} {3,5,11,15} {3,6,17,19}

{3,7,22,24} {3,8,20,27} {3,9,14,16} {3,10,13,18} {4,5,20,25}

{4,6,12,16} {4,7,17,18} {4,8,11,19} {4,9,21,28} {4,10,14,15}

$\{5,6,22,28\}$ $\{5,7,21,27\}$ $\{5,8,12,13\}$ $\{5,9,24,26\}$ $\{5,10,16,17\}$

$\{6,7,20,26\}$ $\{6,8,14,18\}$ $\{6,9,11,13\}$ $\{6,10,25,27\}$ $\{7,8,23,28\}$

$\{7,9,15,19\}$ $\{7,10,11,12\}$ $\{8,9,22,25\}$ $\{8,10,21,24\}$ $\{9,10,20,23\}$

$\{11,14,27,28\}$ $\{11,16,22,23\}$ $\{11,17,24,25\}$ $\{11,18,21,26\}$

$\{12,14,20,24\}$ $\{12,15,26,28\}$ $\{12,18,23,25\}$ $\{12,19,22,27\}$

$\{13,15,21,25\}$ $\{13,16,26,27\}$ $\{13,17,20,28\}$ $\{13,19,23,24\}$

$\{14,17,21,22\}$ $\{14,19,25,26\}$ $\{15,17,23,27\}$ $\{15,18,20,22\}$

$\{16,18,24,28\}$ $\{16,19,20,21\}$

This 4-graph is a block design with parameters $(28,63,9,4,1)$.

Example 7

$\{1,2,19\}$ $\{1,3,6\}$ $\{1,4,30\}$ $\{1,5,11\}$ $\{1,7,28\}$ $\{1,8,23\}$ $\{1,9,21\}$

$\{1,10,17\}$ $\{1,12,20\}$ $\{1,13,24\}$ $\{1,14,15\}$ $\{1,16,25\}$ $\{1,18,31\}$

$\{1,22,26\}$ $\{1,27,29\}$ $\{2,3,20\}$ $\{2,4,7\}$ $\{2,5,31\}$ $\{2,6,12\}$ $\{2,8,29\}$

$\{2,9,24\}$ $\{2,10,22\}$ $\{2,11,18\}$ $\{2,13,21\}$ $\{2,14,25\}$ $\{2,15,16\}$

$\{2,17,26\}$ $\{2,23,27\}$ $\{2,28,30\}$ $\{3,4,21\}$ $\{3,5,8\}$ $\{3,7,13\}$ $\{3,9,30\}$

$\{3,10,25\}$ $\{3,11,23\}$ $\{3,12,19\}$ $\{3,14,22\}$ $\{3,15,26\}$ $\{3,16,17\}$

$\{3,18,27\}$ $\{3,24,28\}$ $\{3,29,31\}$ $\{4,5,22\}$ $\{4,6,9\}$ $\{4,8,14\}$ $\{4,10,31\}$

$\{4,11,26\}$ $\{4,12,24\}$ $\{4,13,20\}$ $\{4,15,23\}$ $\{4,16,27\}$ $\{4,17,18\}$

$\{4,19,28\}$ $\{4,25,29\}$ $\{5,6,23\}$ $\{5,7,10\}$ $\{5,9,15\}$ $\{5,12,27\}$

$\{5,13,25\}$ $\{5,14,21\}$ $\{5,16,24\}$ $\{5,17,28\}$ $\{5,18,19\}$ $\{5,20,29\}$

$\{5,26,30\}$ $\{6,7,24\}$ $\{6,8,11\}$ $\{6,10,16\}$ $\{6,13,28\}$ $\{6,14,26\}$

$\{6,15,22\}$ $\{6,17,25\}$ $\{6,18,29\}$ $\{6,19,20\}$ $\{6,21,30\}$ $\{6,27,31\}$

$\{7,8,25\}$ $\{7,9,12\}$ $\{7,11,17\}$ $\{7,14,29\}$ $\{7,15,27\}$ $\{7,16,23\}$

$\{7,18,26\}$ $\{7,19,30\}$ $\{7,20,21\}$ $\{7,22,31\}$ $\{8,9,26\}$ $\{8,10,13\}$

$\{8,12,18\}$ $\{8,15,30\}$ $\{8,16,28\}$ $\{8,17,24\}$ $\{8,19,27\}$ $\{8,20,31\}$

{8,21,22} {9,10,27} {9,11,14} {9,13,19} {9,16,31} {9,17,29}

{9,18,25} {9,20,28} {9,22,23} {10,11,28} {10,12,15} {10,14,20}

{10,18,30} {10,19,26} {10,21,29} {10,23,24} {11,12,29} {11,13,16}

{11,15,21} {11,19,31} {11,20,27} {11,22,30} {11,24,25} {12,13,30}

{12,14,17} {12,16,22} {12,21,28} {12,23,31} {23,25,26} {13,14,31}

{13,15,18} {13,17,23} {13,22,29} {13,26,27} {14,16,19} {14,18,24}

{14,23,30} {14,27,28} {15,17,20} {15,19,25} {15,24,31} {15,28,29}

{16,18,21} {16,20,26} {16,29,30} {17,19,22} {17,21,27} {17,30,31}

{18,20,23} {18,22,28} {19,21,24} {19,23,29} {20,22,25} {20,24,30}

{21,23,26} {21,25,31} {22,24,27} {23,25,28} {24,26,29} {25,27,30}

{26,28,31}

This 3-graph is a Steiner triple system with parameters (31,155,15,3,1).

Example 8

{1,2,3,4} {1,2,6,7} {1,2,6,9} {1,3,5,7} {1,3,5,9} {1,4,7,8}

{1,4,8,9} {1,5,6,8} {1,5,7,9} {1,6,7,9} {1,7,8,9} {2,3,7,8}

{2,3,8,9} {2,4,5,7} {2,4,5,9} {2,5,6,8} {2,5,7,9} {2,7,8,9}

{3,4,6,7} {3,4,6,9} {3,5,6,8} {3,6,7,9} {3,7,8,9} {4,5,6,8}

{4,5,7,9} {4,6,7,9}

This 4-graph is constructed by considering the 2-graph F_4.

Example 9

{1,2,3,4,8} {1,2,3,4,9} {1,2,3,8,9} {1,2,5,6,7} {1,4,5,8,9}

{1,6,7,8,9} {2,4,6,8,9} {2,5,7,8,9} {3,4,5,6,7} {3,4,7,8,9}

{3,5,6,8,9}

This 5-graph is constructed by adding two new vertices to each edge

of the graph in Example 1, and then adding four new edges. The new edges are chosen by "experimentation".

Example 10

{1,2,3,8,9,10,15} {1,4,5,8,11,12,15} {1,6,7,8,13,14,15}

{2,4,6,9,11,13,15} {2,5,7,9,12,14,15} {3,4,7,10,11,14,15}

{3,5,6,10,12,13,15} {1,2,3,4,5,6,7} {8,9,10,11,12,13,14}

This *7*-graph is constructed by modifying Example 1.

Example 11

{1,2,3,4,5,6,7,8,9,10,22} {1,2,3,4,5,6,7,8,9,10,23}

{1,2,3,8,9,10,15,16,17,22,23} {1,4,5,8,11,12,15,18,19,22,23}

{1,6,7,8,13,14,15,20,21,22,23} {2,4,6,9,11,13,16,18,20,22,23}

{2,5,7,9,12,14,16,19,21,22,23} {3,4,7,10,11,14,17,18,21,22,23}

{3,5,6,10,12,13,17,19,20,22,23} {11,12,13,14,15,16,17,18,19,20,21}

This *11*-graph is constructed by modifying Example 1.

Example 12

{1,2,3} {1,4,5} {1,6,7} {2,4,6} {2,5,7} {3,4,7} {3,5,6} {2,3,4}

{5,6,7}

This *3*-graph is constructed by adding two new edges to the graph in Example 1.

APPENDIX 2

EVALUATION OF THE CHROMATIC NUMBER OF SOME 4-GRAPHS

We present here an argument to show that the 4-graph of Example 5 in Appendix 1 contains a linear $(25,4,3)$-graph as a subgraph.

Let the 4-graph in question be denoted by F. We first point out that F is a block design. Hence the behavior of its vertices is symmetrical. Each vertex is of degree 8 and every pair of vertices meet exactly once.

It is not hard to verify that F is 3-colorable. We shall presently indicate a method to verify that F is not 2-colorable. Hence F contains a 2-critical subgraph G. By deleting all edges containing any chosen vertex, we obtain a subgraph of F which is easily shown to be 2-colorable. It follows that $|\bigcup G|=|\bigcup F|$ and thus G is a linear $(25,4,3)$-graph.

There are 2^{25} functions ψ: $F\to\{1,2\}$. Suppose one of them is a 2-coloring of F. Let $V=\{x\in F:\psi(x)=1\}$ and $\overline{V}=F-V$. We may assume that $|V|\leq 12$. For $1\leq i\leq 3$, let $F_i=\{F\in F:|F\cap V|=i\}$ and $h_i=|F_i|$. We have the following system of equations:

$$h_1+h_2+h_3 = 50$$

$$3h_1+h_2 = \binom{25-|V|}{2}$$

$$h_2+3h_3 = \binom{|V|}{2},$$

since every pair of vertices must meet exactly once. There are three meaningful solutions. We shall deal with each case individually.

(i) $|V|=10,\ h_1=35,\ h_2=0,\ h_3=15$

Let $v \in V$. Now v must meet every other vertex in V exactly once. There are 9 such vertices to meet. These meetings can only be in F_3. Since $|F \cap V|=3$ for $F \in F_3$, v must meet an even number of vertices from V. This is a contradiction.

(ii) $|V|=11,\ h_1=29,\ h_2=4,\ h_3=17$

Since $|\overline{V}|<17$, some $v \in \overline{V}$ must appear at least twice in F_3. Now v must meet every other vertex in $\overline{V}$. These meetings cannot occur in F_3. Since in any edge, v can meet at most two other vertices from $\overline{V}$, v must appear at least seven times in $F_1 \cup F_2$. This is impossible as the degree of v is 8.

(iii) $|V|=12,\ h_1=24,\ h_2=6,\ h_3=20$

This case is quite involved. By the argument of case (ii), no vertex in $\overline{V}$ can appear three times or more in F_3. Since $|\overline{V}|<20$, some $v \in \overline{V}$ must appear twice in F_3. As v must meet every other vertex in $\overline{V}$, v cannot appear in F_2 but must appear six times in F_1.

By the symmetry in behavior of the vertices, we can pick any vertex of F to be v. From the eight edges which contain v, choose any six of them and from each edge choose two vertices other than v. These vertices, together with v, is a candidate for the set $\overline{V}$, and we have shown that $\overline{V}$ must be obtained in this way.

There are $\binom{8}{6}3^6=20412$ candidates for the set $\overline{V}$, a number substantially smaller than 2^{25}. To dispose of these cases, there are many possible approaches, all of which are likely to be tedious. We shall briefly describe the one we adopt.

Let $F_i=\{v,a_i,b_i,c_i\}$, $1\leq i\leq 6$, be the six chosen edges containing v. We shall search for possible candidates for $\overline{V}$ by following a six-level procedure.

We start at level 1 and generate the following pairs of sets by partitioning F_1: $V_1(a)=\{a_1\}$ with $\overline{V_1(a)}=F_1-V_1(a)$, $V_1(b)=\{b_1\}$ with $\overline{V_1(b)}=F_1-V_1(b)$, and $V_1(c)=\{c_1\}$ with $\overline{V_1(c)}=F_1-V_1(c)$.

For each pair of sets at level 1, we proceed to level 2 and generate new pairs of sets by partitioning F_2. Thus the pairs of sets generated from $V_1(a)$ and $V_1(a)$ are: $V_2(aa)=\{a_1,a_2\}$ with $\overline{V_2(aa)}=F_1\cup F_2-V_2(aa)$, $V_2(ab)=\{a_1,b_2\}$ with $\overline{V_2(ab)}=F_1\cup F_2-V_2(ab)$, and $V_2(ac)=\{a_1,c_2\}$ with $\overline{V_2(ac)}=F_1\cup F_2-V_2(ac)$.

For each pair of sets at level 2, we proceed to level 3 and generate new pairs of sets by partitioning F_3. This procedure is repeated until level 6 is processed. This exhausts all possible candidates for $\overline{V}$, $\overline{V_6(abacbb)}$ being an example. As it turns out, none of these candidates has the desired property.

A short-cut is employed beginning at level 2. After a pair of sets is generated, we check if either one contains an edge of F. If this is the case, the pair is discarded and generates no further pairs of sets.

We remark that by a similar but even more involved argument, we can show that the 4-graph of Example 6 in Appendix 1 contains a linear $(28,4,3)$-graph as a subgraph. We shall omit the details.

APPENDIX 3

LOWER BOUNDS FOR $R(k,l;3)$

We list here our lower bounds for the Ramsey numbers $R(k,l;3)$ for $k,l \leq 8$. The previous best values(see [A12], [K3] and [I1]) are given in brackets.

$R(4,6;3)$	$\geq$	27	(21)
$R(4,7;3)$	$\geq$	37	(36)
$R(4,8;3)$	$\geq$	45	(43)
$R(5,5;3)$	$\geq$	44	(36)
$R(5,6;3)$	$\geq$	88	(57)
$R(5,7;3)$	$\geq$	125	(93)
$R(5,8;3)$	$\geq$	199	(136)
$R(6,6;3)$	$\geq$	176	(144)
$R(6,7;3)$	$\geq$	372	(237)
$R(6,8;3)$	$\geq$	571	(373)
$R(7,7;3)$	$\geq$	1018	(474)
$R(7,8;3)$	$\geq$	1727	(847)
$R(8,8;3)$	$\geq$	3625	(1728)

APPENDIX 4

APPLICATIONS OF THE GENERAL COVERING LEMMA

We list here some examples from the literature in which the underlying principle of the General Covering Lemma is used to tackle various problems, although in some of these other devices may be needed as well.

Example 1

The method of the General Covering Lemma seems to have been first applied by G. G. Lorentz([L3]) in connection with a problem in additive number theory. He showed that if $a_1 < a_2 < \ldots$ is any sequence of positive integers, then there exists a sequence $b_1 < b_2 < \ldots$ of integers of density zero such that every sufficiently large integer can be written in the form $a_i + b_j$ for some i and j.

Example 2

Abbott, Liu and Riddell([A9], [A10] and [L1]) made use of the method in obtaining lower bounds for certain van der Waerden numbers and in studying other questions on arithmetic progressions.

Example 3

Erdös([E3] and[E4]) and Herzog and Schönheim([H5]) used

the method in studying property B(that is, 2-colorability) and its generalizations.

Example 4

Chvátal([C2]) used the method in getting good upper bounds for the so-called Turán numbers for hypergraphs.

Example 5

Abbott([A6]) considered the problem of determining the least number of lattice points that one can select from an $n \times n$ square array of lattice points in the plane so that every other lattice point is visible from one of the points selected. The upper bound $c\log n$ was obtained by the method of the General Covering Lemma.

Example 6

It is of interest in coding theory to determine the least number of vertices that may be selected from the set of the vertices of the n-dimensional unit cube so that every vertex of the cube is within a certain specified distance from at least one of the points selected. The best known upper bound for this number is obtained by Ehrenfeuct and Mycielski([E1]) via the method.

Example 7

Rogers([R2]) used a variation of the method, combined with a number of other devices, to get the best known upper bound for

the covering density of n-dimensional Euclidean space by unit spheres.

Many of the above examples are mentioned in a survey paper by Spencer([S4]). See also the paper of Stein([S5]). It is hoped that more applications of the method will be discovered in the future.

APPENDIX 5

A LOWER BOUND FOR $B(5)$

We present here an argument to show that

$$B(5) > 8,$$

that is, the minimal size of 5-graphs without property $B(1,3)$ exceeds 8. Lemma 5.6 of Chapter Five will be appealed to throughout the argument, and explicit references will not be made.

Suppose $\mathcal{F}=\{F_1,\ldots,F_8\}$ is a 5-graph without property $B(1,3)$. We may assume that every pair of vertices of $\mathcal{F}$ meets at least once. It is easily seen that no vertex can be of degree 1, 6, 7 or 8. We consider two cases:

<u>(i) Somce vertex v is of degree 5</u>

Suppose say $v \in F_1 \cap \ldots \cap F_5$. Clearly $F_6 \cap F_7 \cap F_8 = \emptyset$ or we have a $(5,1,3)$-set with two vertices. We assume that

$$|F_6 \cap F_7| \geq |F_6 \cap F_8| \geq |F_7 \cap F_8|$$

and consider four subcases.

<u>(a) $|F_6 \cap F_7| = 1$</u> We have $|\bigcup\mathcal{F}| \geq 13$. In order to meet every other vertex, each vertex must be of degree at least 3. Since v is of degree 5, at least one vertex will be of degree at most 2. This is impossible.

<u>(b) $|F_6 \cap F_7| = 4$</u> Let $F_6 \cap F_7 = \{a,b,c,d\}$. Then v must meet each pair in

$\{a,b,c,d\}\times F_8$. Now v can meet at most four such pairs in each of F_1, $\ldots,F_5$. Hence it must meet exactly four pairs in each edge. It follows that each of $F_1,\ldots,F_5$ contains two vertices from each of $\{a,b,c,d\}$ and F_8. Hence one of a, b, c and d will appear in less than three of $F_1,\ldots,F_5$. Thus v cannot meet all pairs in $\{a,b,c,d\}\times F_8$. This is impossible.

<u>(c) $|F_6 \cap F_7|=2$</u> Let $F_6 \cap F_7=\{a,b\}$. There exist $c\in F_6$ and $d\in F_7$ such that $c,d\notin F_8$. Now v must meet each pair in $\{a,b,c,d\}\times F_8$. This is shown in subcase (b) to be impossible.

<u>(d) $|F_6 \cap F_7|=3$</u> Let $F_6 \cap F_7=\{a,b,c\}$. By subcase (b), $F_6 \cup F_7$ cannot contain a vertex $d\notin F_8$. Suppose $F_1\cup\ldots\cup F_5$ contains a vertex $d\notin F_8$. In order that d should meet every other vertex, it must be of degree at least 3, say $d\in F_1\cap F_2\cap F_3$. Now v must meet each pair in $\{a,b,c\}\times F_8$. It can meet at most two such pairs in each of F_1, F_2 and F_3, and four such pairs in each of F_4 and F_5. This is impossible.

Hence $|\bigcup F|=9$. Now the sum of the degrees of the vertices is 40 while the degree of each vertex lies between 2 and 5. There are five possible decompositions of 40 into 9 such parts:

$$\begin{aligned} 40 &= 5+5+5+5+5+5+5+3+2 \\ &= 5+5+5+5+5+5+4+4+2 \\ &= 5+5+5+5+5+5+4+3+3 \\ &= 5+5+5+5+5+4+4+4+3 \\ &= 5+5+5+5+4+4+4+4+4. \end{aligned}$$

Each possibility can be disposed fairly easily.

(ii) No vertex is of degree 5

Clearly $|\bigcup F|\geq 10$. A vertex of degree 2 cannot meet every other vertex. Hence all vertices are of degree 3 or 4. Let A and B denote respectively the sets of vertices of degree 4 and 3. Let $a=|A|$ and $b=|B|$. Then $4a+3b=40$. We consider the three possible subcases.

(a) $a=10$, $b=0$ Every pair of vertices must meet at least twice. Since each vertex is only of degree 4, this is impossible.

(b) $a=7$, $b=4$ Each vertex in A must meet every other vertex in A at least twice, and every vertex in B at least once. Hence it meets every other vertex in A exactly twice and every vertex in B exactly once. Let $h_0,\ldots,h_4$ denote the number of edges of F which contain respectively $0,\ldots,4$ vertices in B. Then we have the following system of equations:

$$\begin{aligned}
5h_0+4h_1+3h_2+2h_3+h_4 &= 28\\
h_1+2h_2+3h_3+4h_4 &= 12\\
10h_0+6h_1+3h_2+h_3 &= 42\\
h_2+3h_3+6h_4 &= 10\\
4h_1+6h_2+6h_3+4h_4 &= 28.
\end{aligned}$$

There are three meaningful solutions:

$$(h_0,h_1,h_2,h_3,h_4) = (3,1,1,3,0)$$

$$(h_0,h_1,h_2,h_3,h_4) = (2,3,1,1,1)$$

$$(h_0,h_1,h_2,h_3,h_4) = (3,0,4,0,1).$$

Each possibility can be disposed fairly easily.

(c) $a=4$, $b=8$ Each vertex in B must meet every other vertex at least once. Hence it meets exactly one vertex twice. Let $B=\{a,b,c,d,e,f,g,h\}$.

Suppose $a,b\in F_1\cap F_2$. Now each of a and b must appear separately in one other edge, say $a\in F_3$ and $b\in F_4$. Clearly $F_3-\{a\}=F_4-\{b\}$. Furthermore, we must have $F_3\cap F_4\subset A$. We may assume that $F_1=\{a,b,c,d,e\}$ and $F_2=\{a,b,f,g,h\}$. Now each of c, d and e must appear in two of $F_5,\ldots,F_8$. Hence one of them, say c, will either meet d or e three times or both d and e twice. Neither case is possible. Therefore every vertex in B meets every other vertex in B exactly once. It follows that every vertex in A also meets every other vertex in A exactly once. Let $h_1,\ldots,h_5$ denote the number of edges of F which contain respectively $1,\ldots,5$ vertices in B. Then we have the following system of equations:

$$4h_1+3h_2+2h_3+h_4 = 16$$

$$h_1+2h_2+3h_3+4h_4+5h_5 = 24$$

$$6h_1+3h_2+h_3 = 12$$

$$h_2+3h_3+6h_4+10h_5 = 28$$

$$4h_1+6h_2+6h_3+4h_4 = 20.$$

There are four meaningful solutions:

$$(h_1,h_2,h_3,h_4,h_5) = (1,0,6,0,1)$$

$$(h_1,h_2,h_3,h_4,h_5) = (1,1,3,3,0)$$

$$(h_1,h_2,h_3,h_4,h_5) = (0,3,3,1,1)$$

$$(h_1,h_2,h_3,h_4,h_5) = (0,4,0,4,0).$$

Each possibility can be disposed fairly easily.

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